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Mechanics of a Lamina

Book Geoff Eckold, Chapter 3, pp 49-65

A **Laminate** is consisting of several **Laminas or Plies or Layers**.

A **Lamina** is consisting of Fibers and Matrix.

Micromechanics (in μm -mm range) is dealing for example with the determination of **Lamina** constitutive properties from those of Fiber and Matrix, Fiber-Matrix interface stresses, etc.

Assumptions:

- Linear Elasticity: Matrix and Fiber behave as linear elastic material (viscoelasticity of Matrix: see previous chapter)
- Perfect bond, no strain discontinuity across interface
- Fibers are arranged in a regular or repeating array

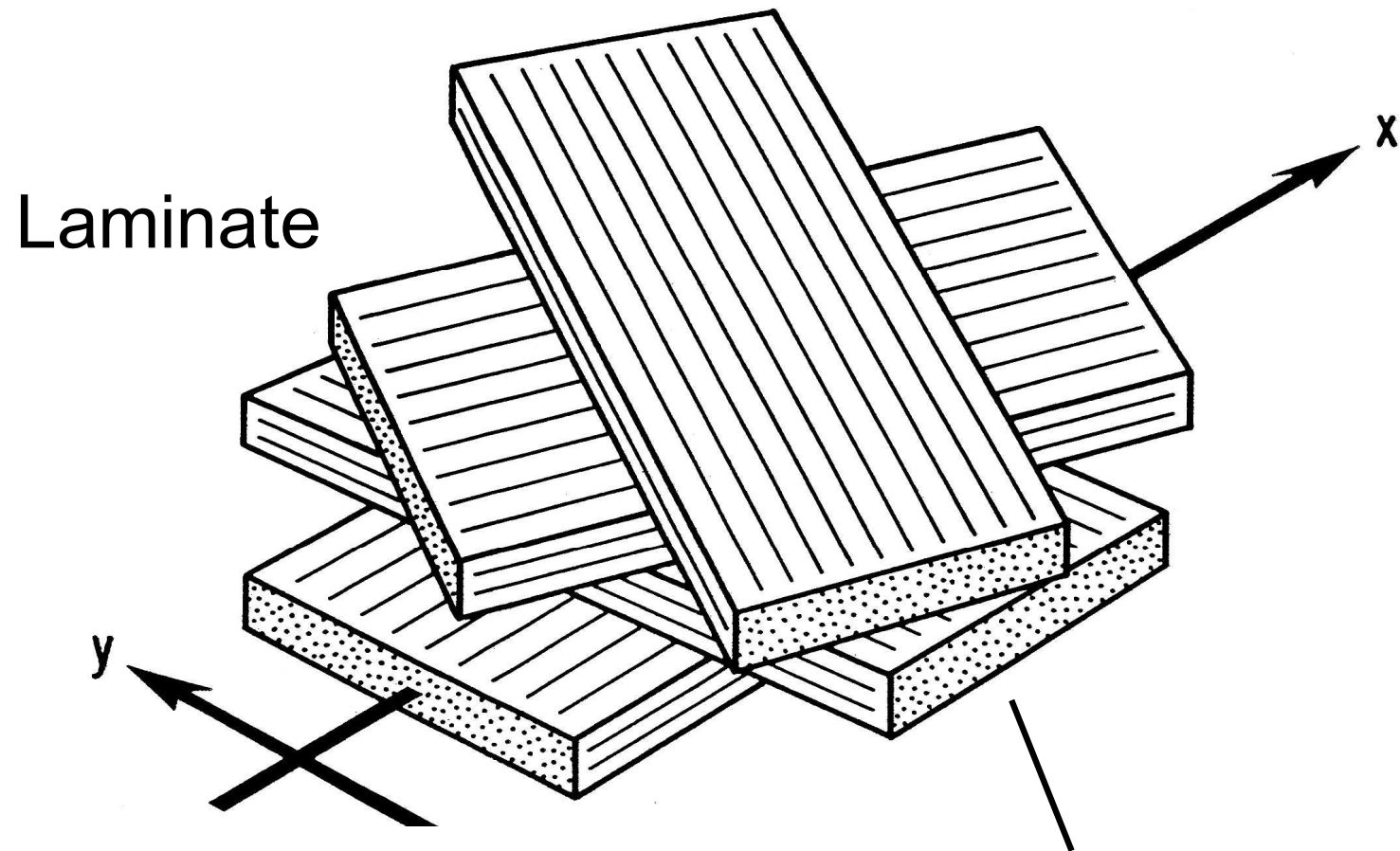
Functional requirements for Fibers:

- High E-Modulus
- High ultimate strength
- Low variation between individual fibers
- Retain the strength during handling and fabrication
- Uniform diameter and surface

Functional requirements for Matrix:

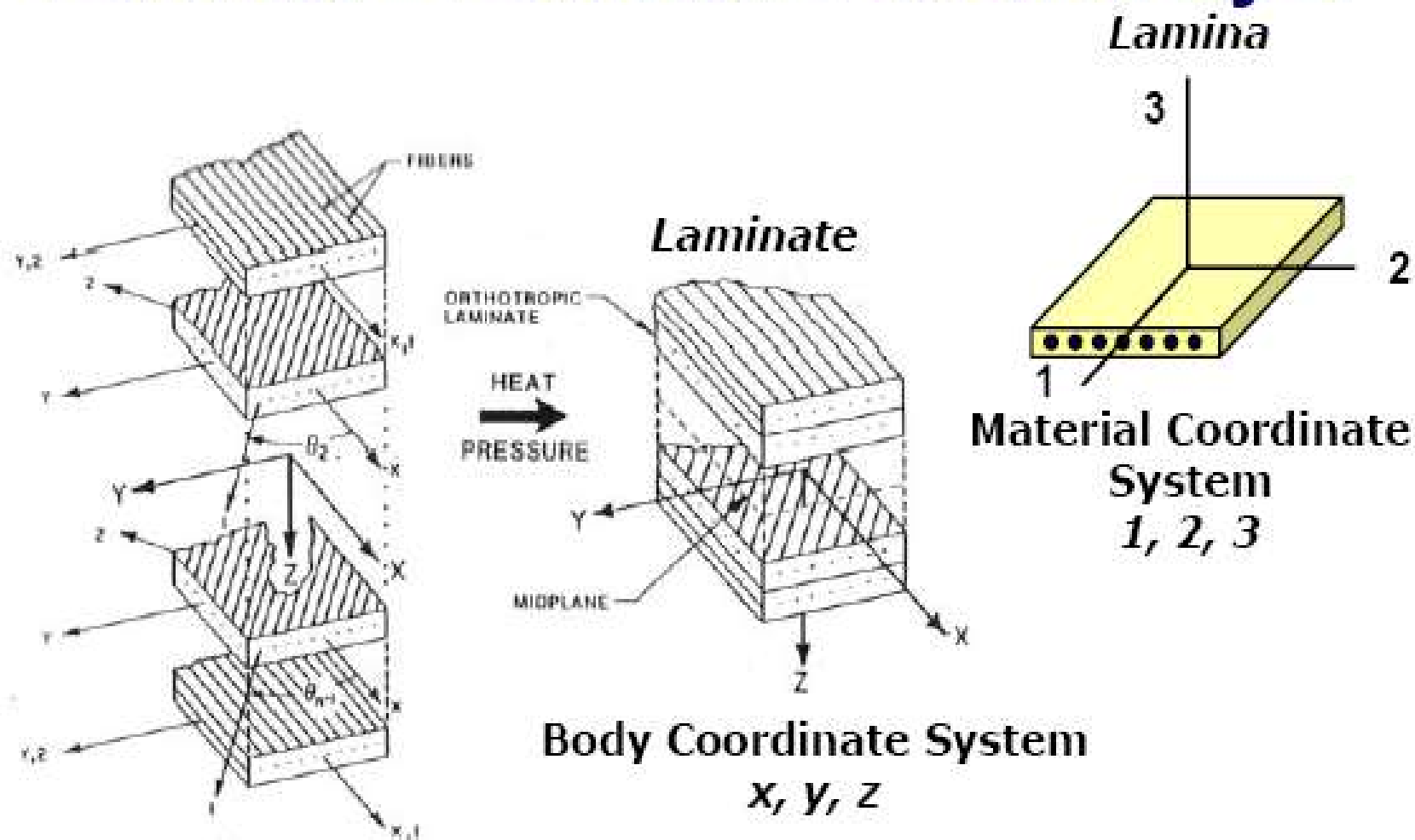
- Bind together the fibers and protect their surfaces
- Transfer stresses to the fibers efficiently
- Chemically compatible with fibers over a long period
- Thermally compatible with fibers

Modell of a Laminate



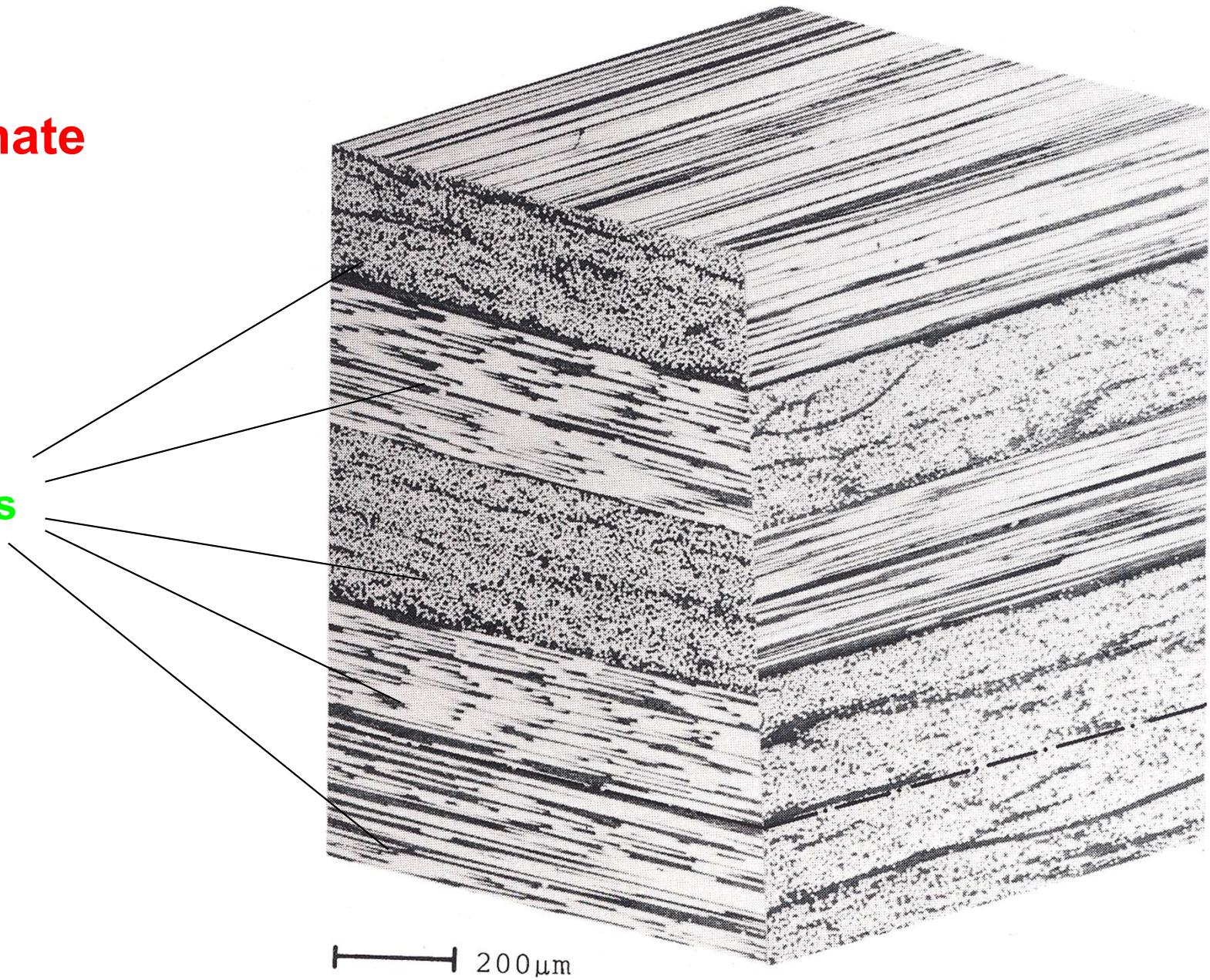
**A Unidirectionally Reinforced Lamina
(UD)**

Lamina – Laminate Coord. Sys.

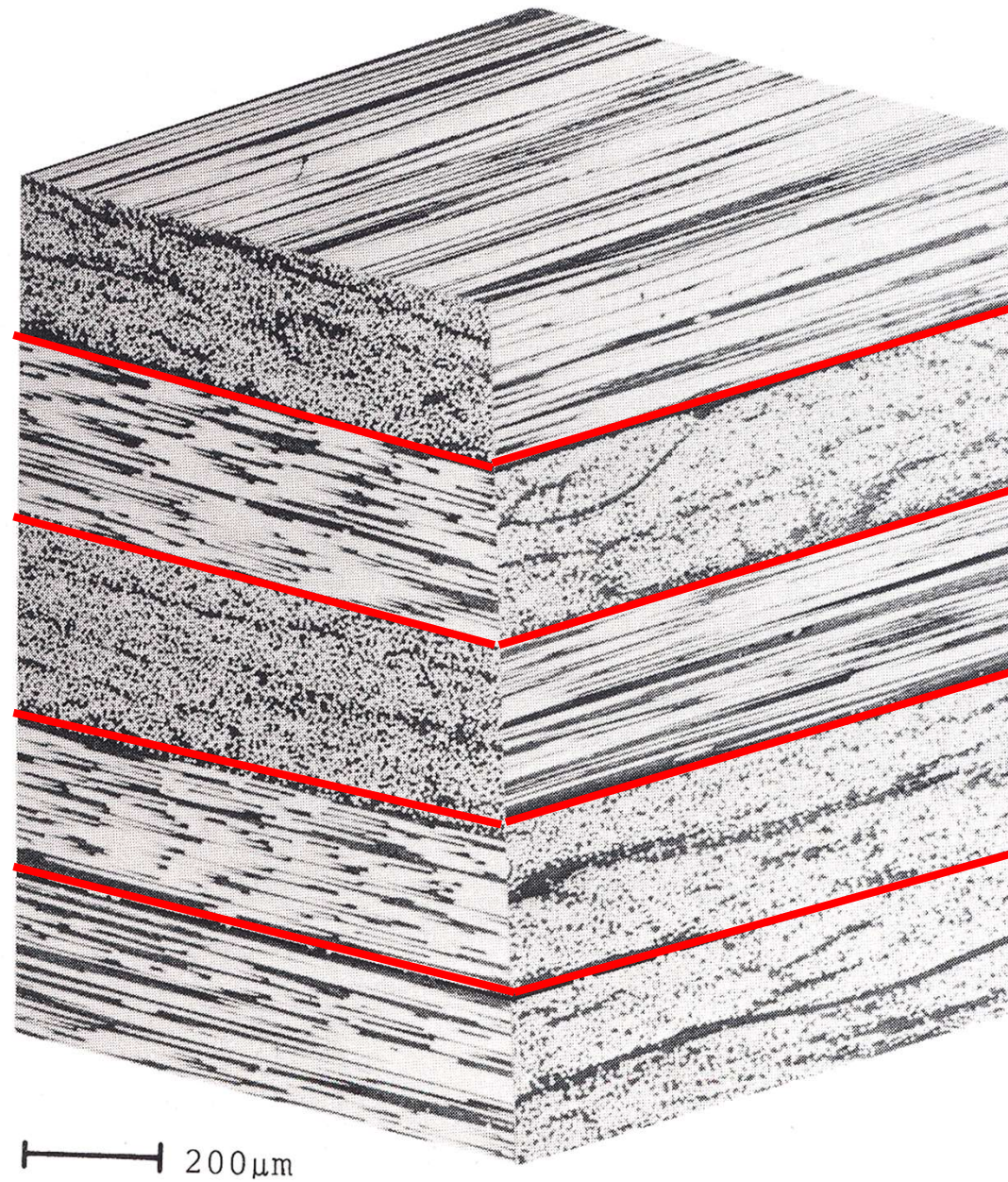


A Laminate

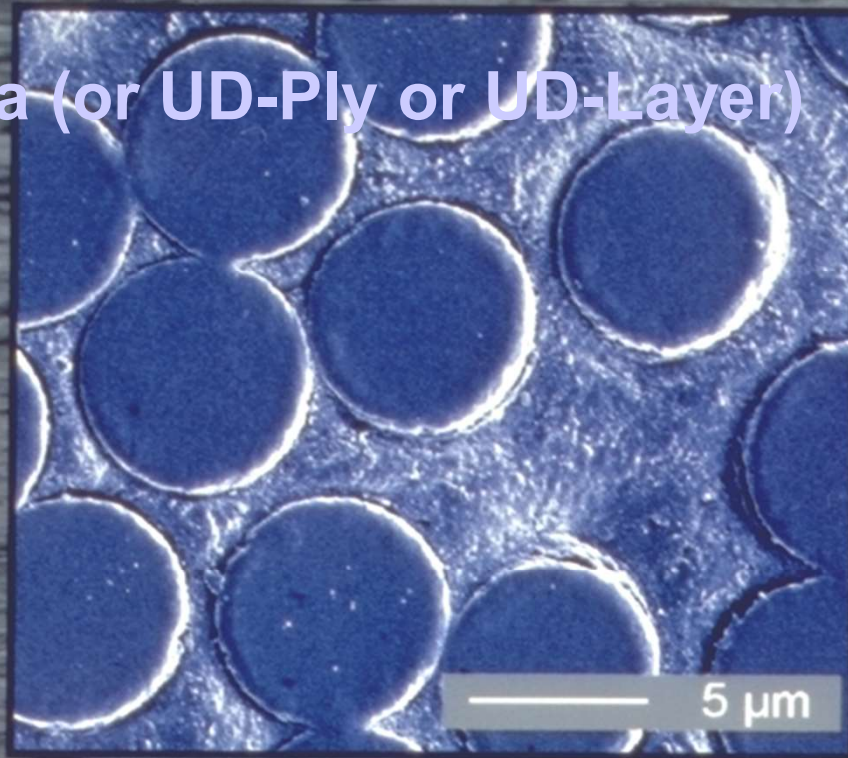
UD-Laminas



A Laminate



UD-Lamina (or UD-Ply or UD-Layer)

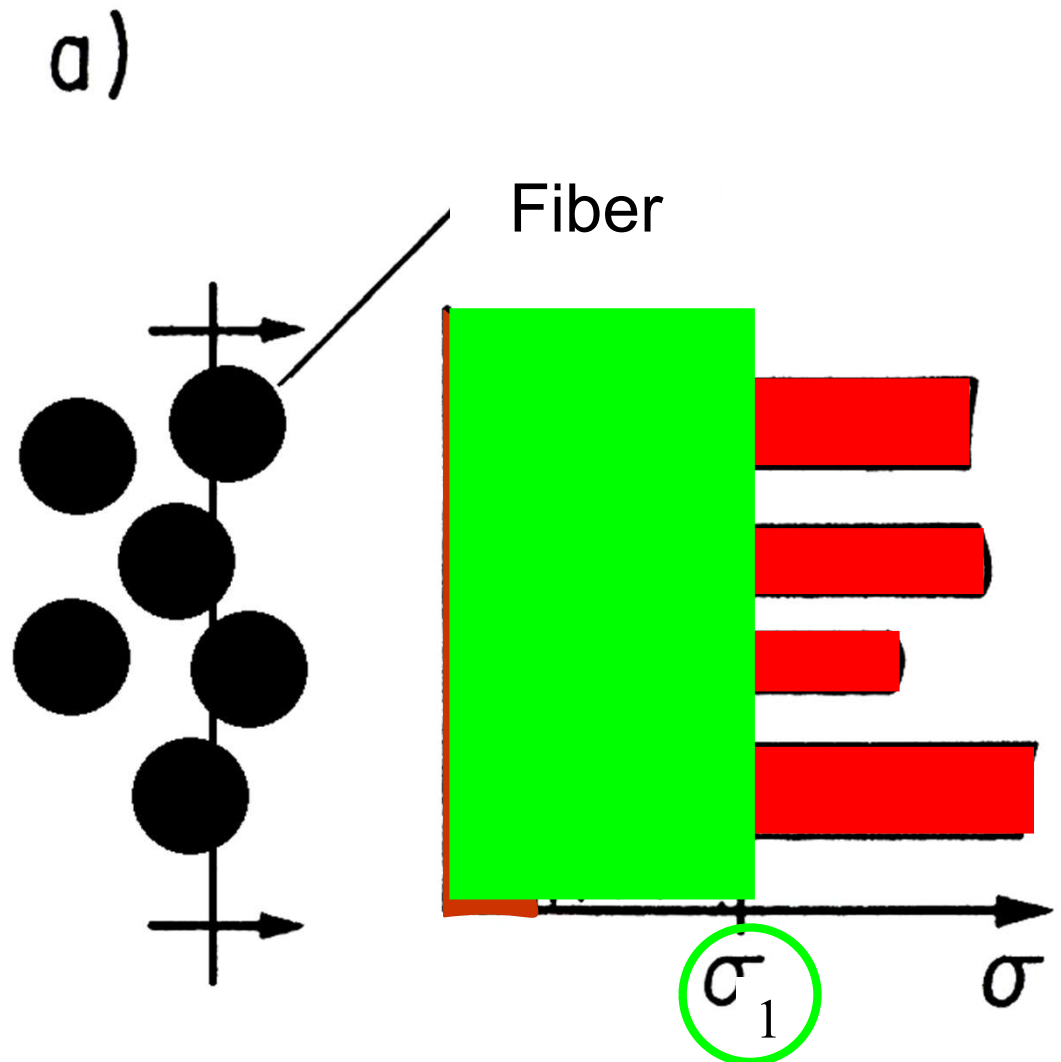
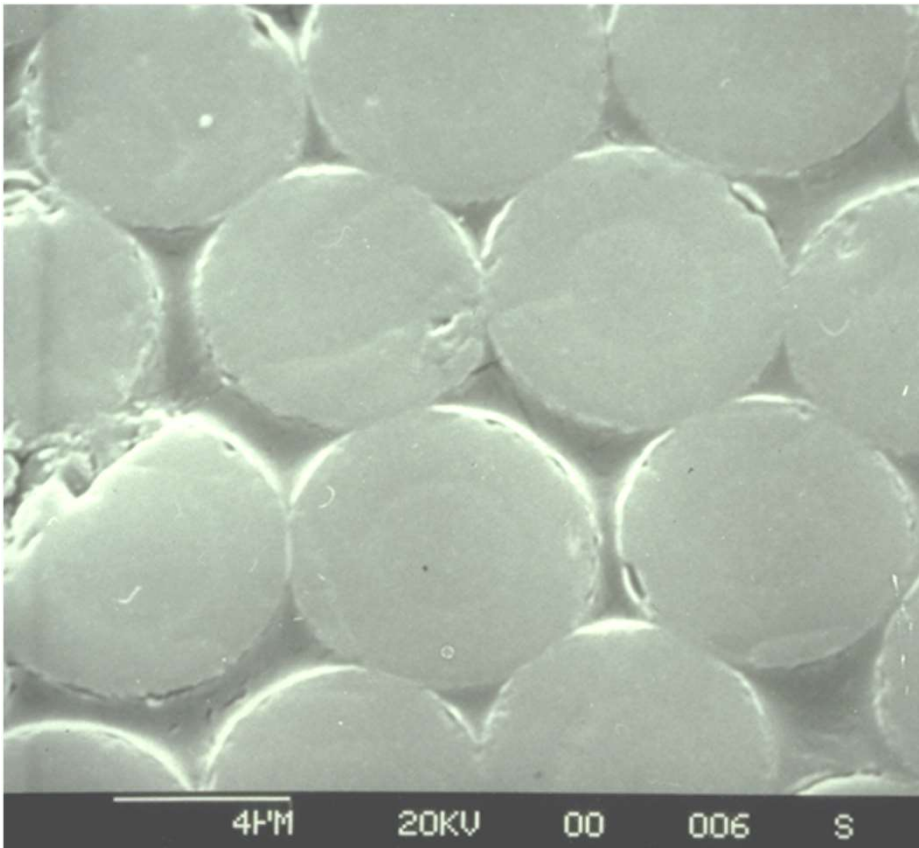


300 μm

Definitions:

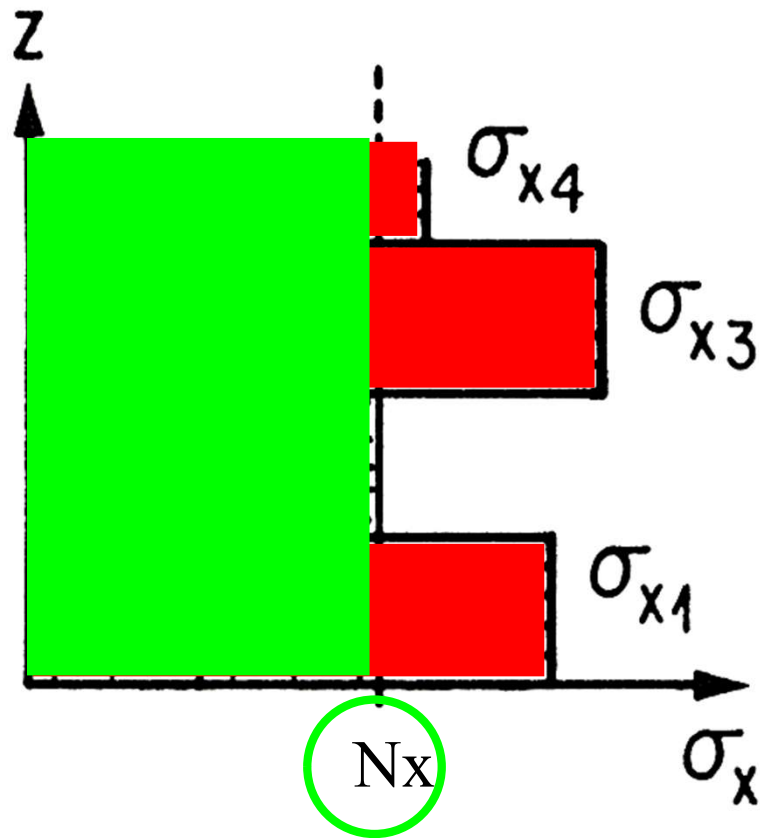
- **Homogeneous:** Properties are not function of the position of the material points
- **Isotropy:** Properties are not function of the orientation.
2 independent material constants: E and ν
- **Anisotropy:** Properties are function of the orientation with no planes of symmetry.
21 independent material constants

Micromechanics



Macromechanics

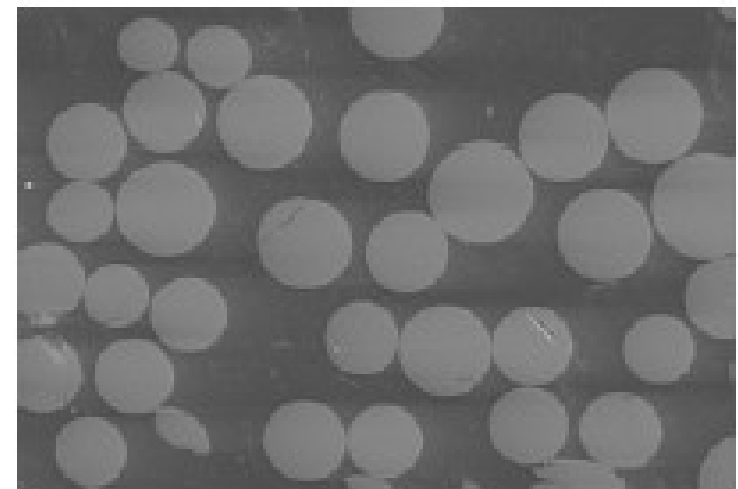
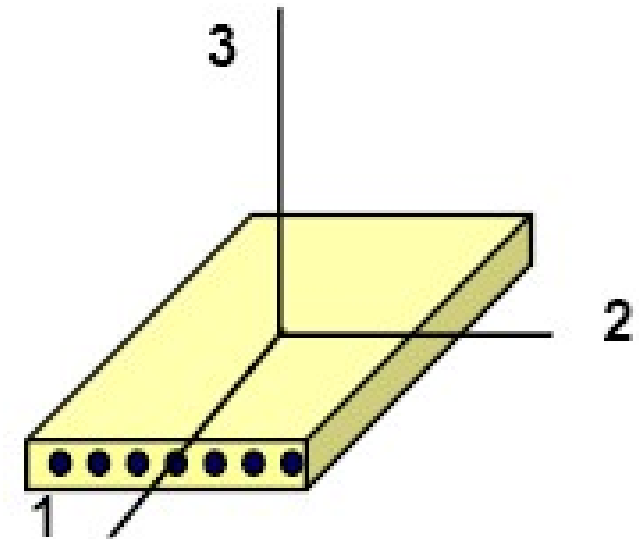
b)



Equivalent Homogeneous Material

Important Assumption:

- We assume that the fiber and matrix properties can be smeared and represented as an equivalent homogenous material with orthotropic material properties
- This allows us to develop the stress strain behavior of the material making the structural level response tractable
- Otherwise we would have to deal with micromechanics



Unidirectional Lamina

(UD-Lamina)

Stiffness of a UD-Lamina

Orthotropic Lamina – Symmetric Compliance Matrix

Due to the 3 reciprocal relations, only 9 independent elastic constants are needed

$$\begin{aligned}
 S_{11} &= \frac{1}{E_{11}} & S_{12} &= \frac{-\nu_{12}}{E_{11}} & S_{13} &= \frac{-\nu_{13}}{E_{11}} \\
 S_{22} &= \frac{1}{E_{22}} & S_{23} &= \frac{-\nu_{23}}{E_{22}} & S_{33} &= \frac{1}{E_{33}} \\
 S_{44} &= \frac{1}{G_{23}} & S_{55} &= \frac{1}{G_{13}} & S_{66} &= \frac{1}{G_{12}}
 \end{aligned}
 \quad
 \begin{Bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}
 =
 \begin{bmatrix}
 S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\
 S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\
 S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\
 0 & 0 & 0 & S_{44} & 0 & 0 \\
 0 & 0 & 0 & 0 & S_{55} & 0 \\
 0 & 0 & 0 & 0 & 0 & S_{66}
 \end{bmatrix}
 \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix}$$

Zero entries in the upper right and lower left portions of the compliance matrix characterize orthotropic behavior

Orthotropic Lamina – Symmetric Stiffness Matrix

The inverse relations are...

$$\begin{aligned} C_{11} &= \frac{S_{22}S_{33} - S_{23}S_{23}}{S} & C_{12} &= \frac{S_{13}S_{23} - S_{12}S_{33}}{S} \\ C_{22} &= \frac{S_{33}S_{11} - S_{13}S_{13}}{S} & C_{13} &= \frac{S_{12}S_{23} - S_{13}S_{22}}{S} \\ C_{33} &= \frac{S_{11}S_{22} - S_{12}S_{12}}{S} & C_{23} &= \frac{S_{12}S_{13} - S_{23}S_{11}}{S} \\ C_{44} &= \frac{1}{S_{44}} & C_{55} &= \frac{1}{S_{55}} & C_{66} &= \frac{1}{S_{66}} \end{aligned}$$

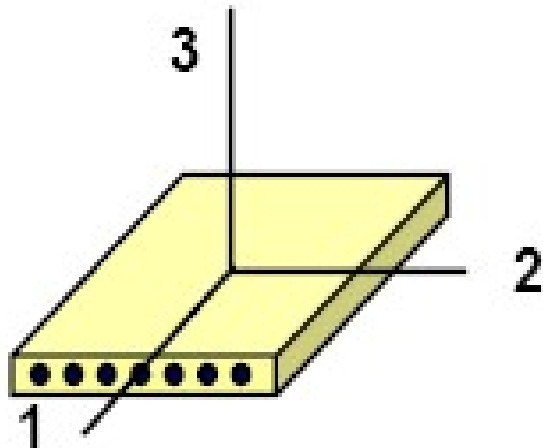
$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{Bmatrix}$$

where

$$S = S_{11}S_{22}S_{33} - S_{11}S_{23}S_{23} - S_{22}S_{13}S_{13} - S_{33}S_{12}S_{12} + 2S_{12}S_{23}S_{13}$$

Transversely Isotropic

Between orthotropic material behavior and isotropic material behavior is **transversely isotropic** behavior. We assume properties in the 2 and 3 directions are similar. We assume that...



$$E_{22} = E_{33} \quad \nu_{12} = \nu_{13} \quad G_{12} = G_{13}$$

$$\text{and } G_{23} = \frac{E_{22}}{2(1 + \nu_{23})}$$

Transversely Isotropic

If we assume that properties in the 2 and 3 directions are similar, we arrive at the compliance matrix...

$$S_{11} = \frac{1}{E_{11}} \quad S_{12} = \frac{-\nu_{12}}{E_{11}} \quad S_{22} = \frac{1}{E_{22}} \quad S_{23} = \frac{-\nu_{23}}{E_{22}}$$

$$S_{33} = \frac{1}{E_{33}} \quad S_{44} = \frac{1}{G_{23}} = \frac{2(1+\nu_{23})}{E_{22}} \quad S_{55} = \frac{1}{G_{12}}$$

$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix}$$

We now have 5 independent material properties

$$E_{11}, E_{22}, \nu_{12}, \nu_{23}, G_{12}$$

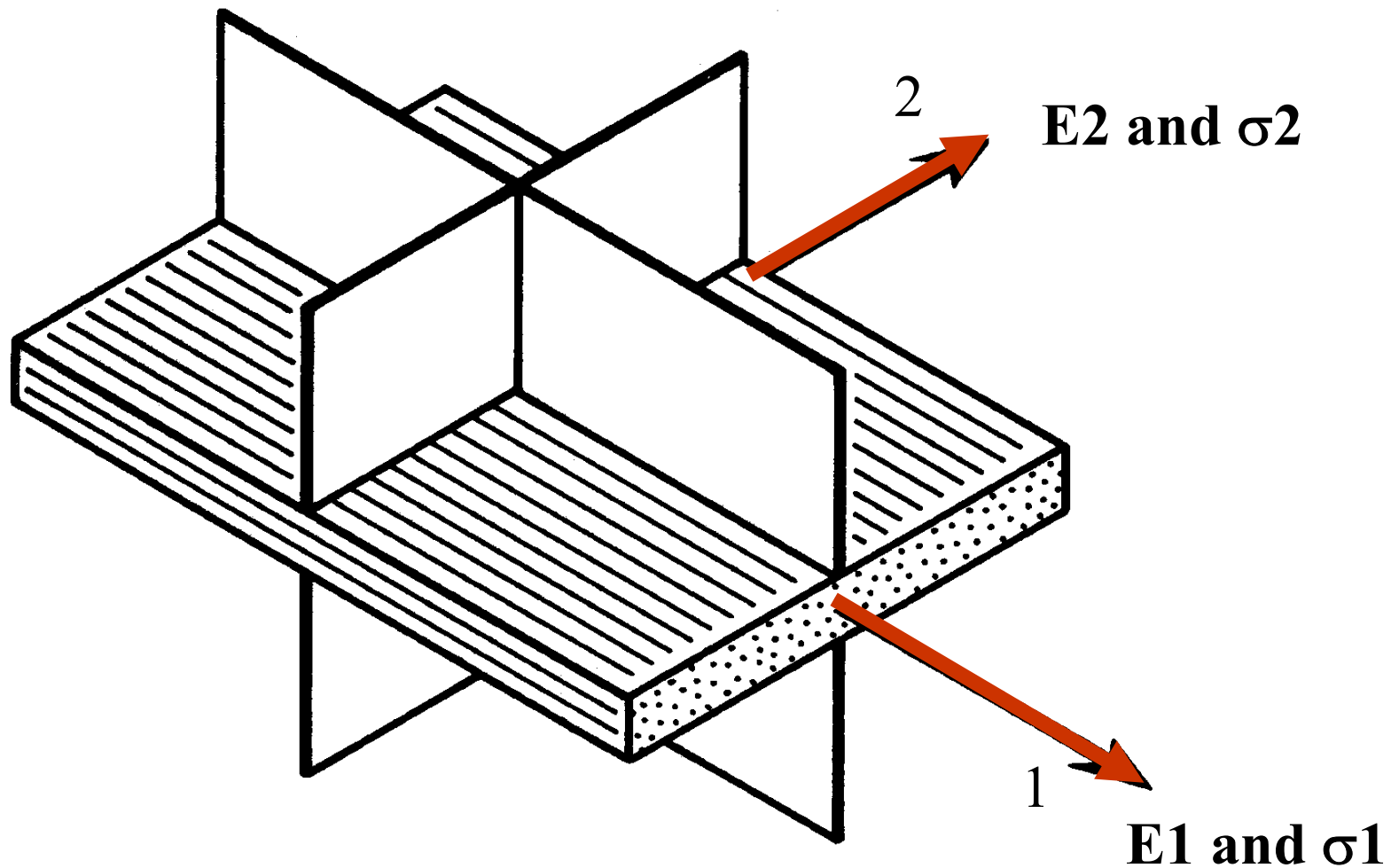
Transversely Isotropic

If we assume that properties in the 2 and 3 directions are similar, we arrive at the stiffness matrix...

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}$$

Transversely Isotropic: a material which contains a plane in which the mechanical properties are equal in all directions

Symmetrical planes of a transverse isotrop UD-Lamina



Plane Stress: Out-of-Plane Strains

$$\sigma_{33} = \tau_{13} = \tau_{23} = 0$$

As a result the out-of-plane strains are...

$$\gamma_{13} = \gamma_{23} = 0$$

$$\varepsilon_{33} = S_{13}\sigma_{11} + S_{23}\sigma_{22}$$

Note that the out-of-plane normal strain ε_{33} **is not zero**, as a result of the Poisson's ratios ν_{13} , ν_{23} , acting through the S_{13} & S_{23} !

Reduced Compliance Matrix

The Reduced Compliance Matrix is a result of our assumption of transverse isotropy and plane stress...

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix}$$

Transversely Isotropic

$$S_{11} = \frac{1}{E_{11}} \quad S_{12} = \frac{-\nu_{12}}{E_{11}} = \frac{-\nu_{21}}{E_2}$$

$$S_{22} = \frac{1}{E_{22}} \quad S_{66} = \frac{1}{G_{12}}$$

4 independent material properties

$$E_{11}, E_{22}, \nu_{12}, G_{12}$$

Why ??

Isotropic

$$S_{11} = S_{22} = \frac{1}{E} \quad S_{12} = \frac{-\nu}{E}$$

$$S_{66} = \frac{1}{G} = \frac{2(1+\nu)}{E}$$

2 independent material properties

$$E, \nu \text{ or } G, \nu \text{ or } E, G$$

Strain-Stress relation of a UD-Lamina, Plane stress

$$\varepsilon_1 = \frac{1}{E_1} \sigma_1 - \frac{\nu_{21}}{E_2} \sigma_2$$

$$\varepsilon_2 = -\frac{\nu_{12}}{E_1} \sigma_1 + \frac{1}{E_2} \sigma_2$$

$$\gamma_{12} = \frac{1}{G_{12}} \tau_{12}$$

Stress-Strain relation of a UD-Lamina, Plane stress

Inversion of the Compliance Matrix

$$\sigma_1 = \frac{E_1}{1 - \nu_{21}\nu_{12}} \varepsilon_1 + \frac{\nu_{12}E_2}{1 - \nu_{21}\nu_{12}} \varepsilon_2$$

$$\sigma_2 = \frac{\nu_{21}E_1}{1 - \nu_{21}\nu_{12}} \varepsilon_1 + \frac{E_2}{1 - \nu_{21}\nu_{12}} \varepsilon_2$$

$$\tau_{12} = G_{12} \gamma_{12}$$

Stiffness Matrix Q of a UD-Lamina

	ε_1	ε_2	γ_{12}
σ_1	Q_{11}	Q_{12}	
σ_2	Q_{21}	Q_{22}	
τ_{12}			Q_{66}

Stiffnesses Q_{ij} of a UD-Lamina

$$Q_{11} = \frac{E_1}{1 - \nu_{21}\nu_{12}}; Q_{22} = \frac{E_2}{1 - \nu_{21}\nu_{12}}; Q_{66} = G_{12}$$
$$Q_{21} = \frac{\nu_{21}E_1}{1 - \nu_{21}\nu_{12}}; Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{21}\nu_{12}}$$

Symmetry ?

Stiffness matrix Q:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \left(\frac{N}{mm^2} \right)$$

Compliance matrix S:

Symmetry ?

$$S = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \left(\frac{mm^2}{N} \right)$$

Symmetry ??

$$Q_{12} = Q_{21}$$

and

$$S_{12} = S_{21}$$

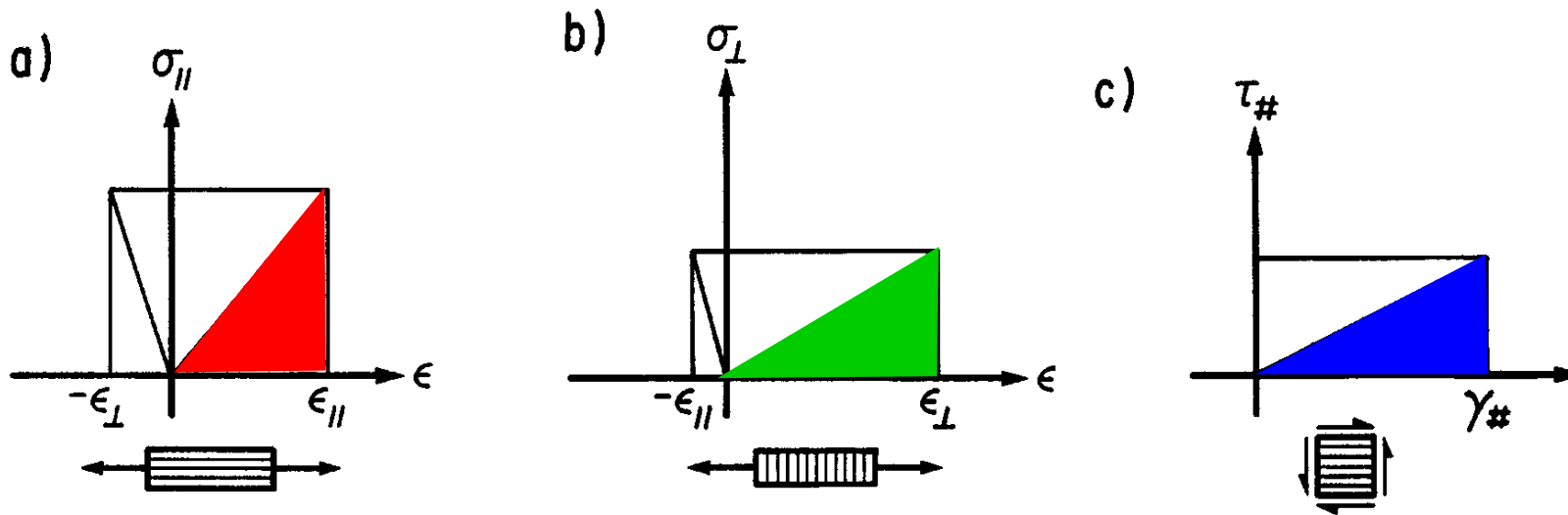
???

Coupling Terms

Elastic Energy

The stored elastic energy in the UD-Lamina is:

$$W = \frac{1}{2} \left[\underline{\sigma_1 \varepsilon_1} + \underline{\sigma_2 \varepsilon_2} + \underline{\tau_{12} \gamma_{12}} \right]$$



Replace the strains with stresses using the compliance matrix as follow:

	σ_1	σ_2	τ_{12}
ε_1	S_{11}	S_{12}	
ε_2	S_{21}	S_{22}	
γ_{12}			S_{66}

We obtain :

$$W = \frac{1}{2} \left[S_{11} \sigma_1^2 + (S_{12} + S_{21}) \sigma_1 \sigma_2 + S_{22} \sigma_2^2 + S_{66} \tau_{12}^2 \right]$$

Partial differentiation provides the following strain-stress relation:

$$\frac{\partial W}{\partial \sigma_1} = \left[S_{11}\sigma_1 + \frac{1}{2}(S_{12} + S_{21})\sigma_2 \right] = \varepsilon_1$$

$$\frac{\partial W}{\partial \sigma_2} = \left[\frac{1}{2}(S_{12} + S_{21})\sigma_1 + S_{22}\sigma_2 \right] = \varepsilon_2$$

Compare the equations with the strain-stress relation through compliance matrix

	σ_1	σ_2	τ_{12}
ε_1	S_{11}	S_{12}	
ε_2	S_{21}	S_{22}	
γ_{12}			S_{66}

we obtain:

$$S_{12} = S_{21}$$

If we use the stiffness matrix

	ε_1	ε_2	γ_{12}
σ_1	Q_{11}	Q_{12}	
σ_2	Q_{21}	Q_{22}	
τ_{12}			Q_{66}

and replace stresses with strains, we obtain in a similar way the following equation:

$$Q_{12} = Q_{21}$$

or following equation is obtained for engineering constants:

$$\nu_{21}E_1 = \nu_{12}E_2$$

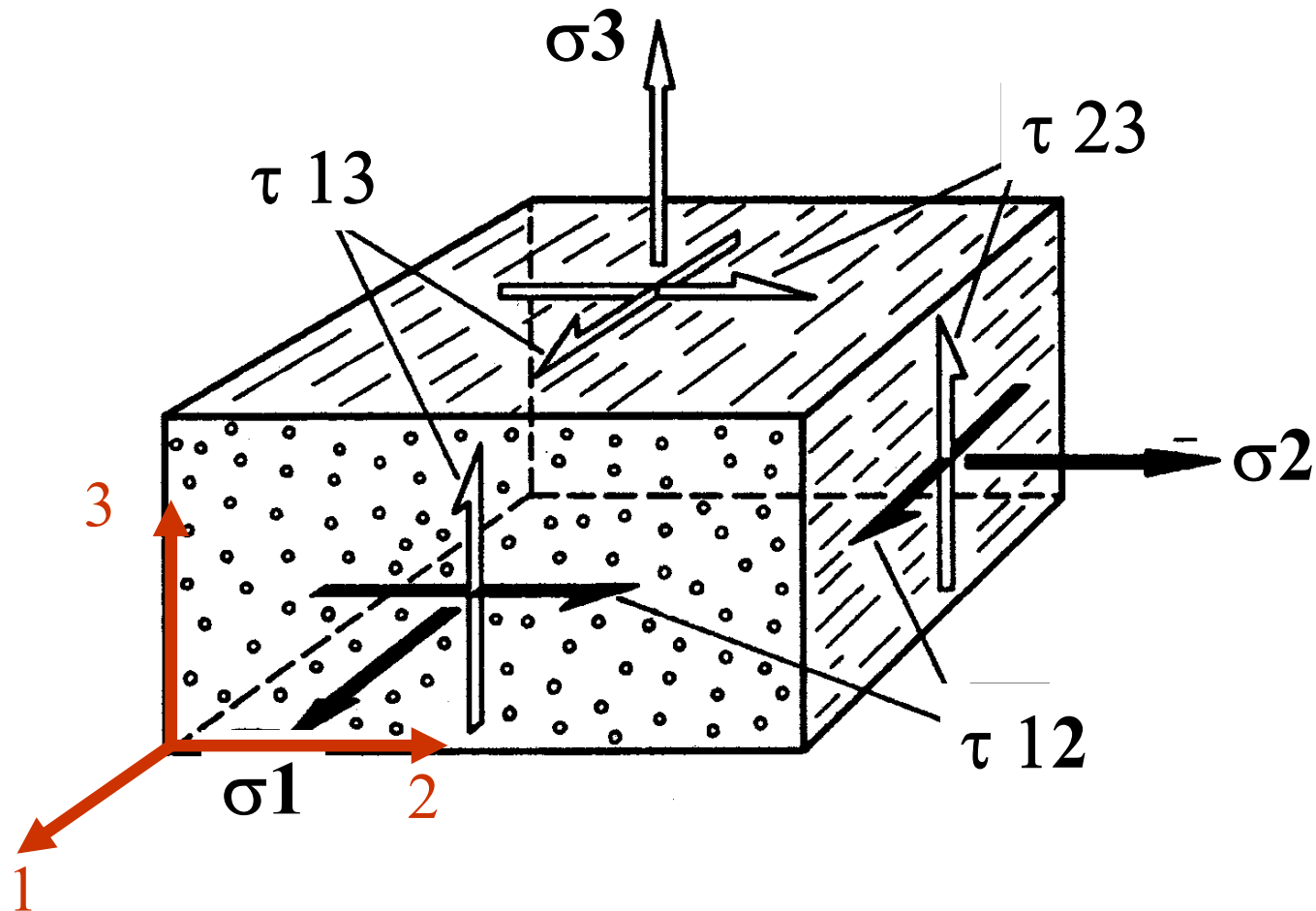
or:

$$\frac{\nu_{21}}{\nu_{12}} = \frac{E_2}{E_1}$$

Definitions:

- **Orthotropy:** Anisotropy with 3 orthogonal planes of symmetry.
9 independent constants
- **Transverse Isotropy:** Orthotropy with a plane at which there is Isotropy
5 independent constants
- **Transverse Isotropy and Plane Stress:**
4 independent constants

Stress State of a Unidirectional (UD) Lamina: Transverse Isotropy, Homogeneous



Plane Stress: σ_1 σ_2 τ_{12}

Independent Elasticity Constants

If there are more symmetry conditions, there will be further reductions in the number of constants. For a **cross-ply** laminate with $E_1=E_2$, there are **3** and for an **isotropic** material (for example mat-laminate with randomly distributed fibers) **2** independent constants.

Mechanics of Materials Approaches to Elastic Property Estimates

Elasticity constants of a UD-Lamina are dependent on the following

E_F = **Fiber E-Modulus**

ν_F = **Fiber Poisson's Ratio**

E_M = **Matrix E-Modulus**

ν_M = **Matrix Poisson's Ratio**

ϕ = **Fiber Volume Fraction**

Rule of Mixtures



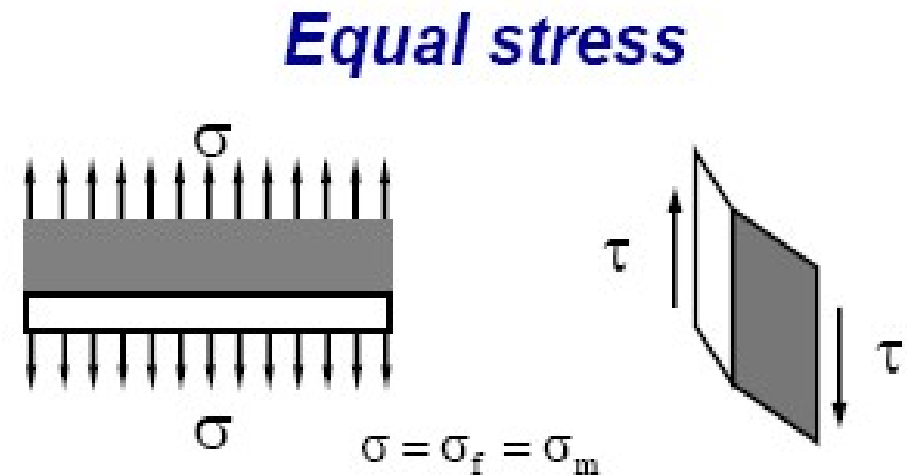
$$\varepsilon = \varepsilon_f = \varepsilon_m$$

$$\begin{aligned}\sigma &= \sigma_f V_f + \sigma_m (1 - V_f) \\ &= E_f \varepsilon V_f + E_m \varepsilon (1 - V_f)\end{aligned}$$

$$E_{11} = E_1^f V^f + E_m (1 - V^f)$$

Similar for ν_{12}

$$\nu_{12} = \nu_{12}^f V^f + \nu_m (1 - V^f)$$



$$\sigma = \sigma_f = \sigma_m$$

$$\varepsilon = \varepsilon_f V_f + \varepsilon_m (1 - V_f)$$

$$\varepsilon = \frac{\sigma_f}{E_f} V_f + \frac{\sigma_m}{E_m} (1 - V_f)$$

$$\frac{1}{E_{22}} = \frac{V^f}{E_2^f} + \frac{(1 - V^f)}{E^m} \quad \frac{1}{G_{12}} = \frac{V^f}{G_{12}^f} + \frac{(1 - V^f)}{G^m}$$

Not so good for E_2 & G_{12}

The longitudinal E-Modulus (parallel to the fiber direction) can be derived from the following so called rule of mixture:

$$E_1 = \Phi_F E_F + (1 - \Phi_F) E_M$$

where ϕ_F = fiber volume content of the UD-Lamina

The major poisson's ratio ν_{12} caused by longitudinal stresses following the rule of mixture is:

$$\nu_{12} = \Phi_F \nu_F + (1 - \Phi_F) \nu_M$$

And the minor poisson's ratio is:

$$\nu_{21} = \nu_{12} \frac{E_2}{E_1}$$

Semi Empirical Equations Based on Experiments

The transverse E-modulus for **isotropic** fibers according to ,Puck' can be obtained:

$$E_2 = E_M^o \frac{1 + 0.85\Phi_F^2}{\Phi_F E_M^o / E_F + (1 - \Phi_F)^{1.25}}$$

$$E_M^o = \frac{E_M}{1 - \nu_M^2}$$

According to ,Puck' for **isotropic** fibers:

$$G_{12} = G_M \frac{\left(1 + 0.60\Phi_F^{0.5}\right)}{\Phi_F G_M / G_F + \left(1 - \Phi_F\right)^{1.25}}$$

According to ,Förster' and ,Schneider' for **isotropic** fibers:

$$E_2 = E_M^o \frac{1}{\Phi_F E_M^o / E_F + (1 - \Phi_F)^{1.45}}$$

$$E_M^o = \frac{E_M}{1 - \nu_M^2}$$

According to ,Förster' and ,Schneider' for **isotropic** fibers:

$$G_{12} = G_M \frac{\left(1 + 0.4\Phi_F^{0.5}\right)}{\Phi_F G_M / G_F + \left(1 - \Phi_F\right)^{1.45}}$$

Following ,Tsai' for **isotropic** fibers:

$$E_2 = \frac{E_M (1 + \xi \eta \Phi_F)}{1 - \eta \Phi_F} \quad \text{where}$$

$$\eta = \frac{E_F / E_M - 1}{E_F / E_M + \xi} \quad \text{and} \quad \xi = 2$$

Following ,Tsai' for **isotropic** fibers:

$$G_{12} = \frac{G_M (1 + \xi \eta \Phi_F)}{1 - \eta \Phi_F} \quad \text{where}$$

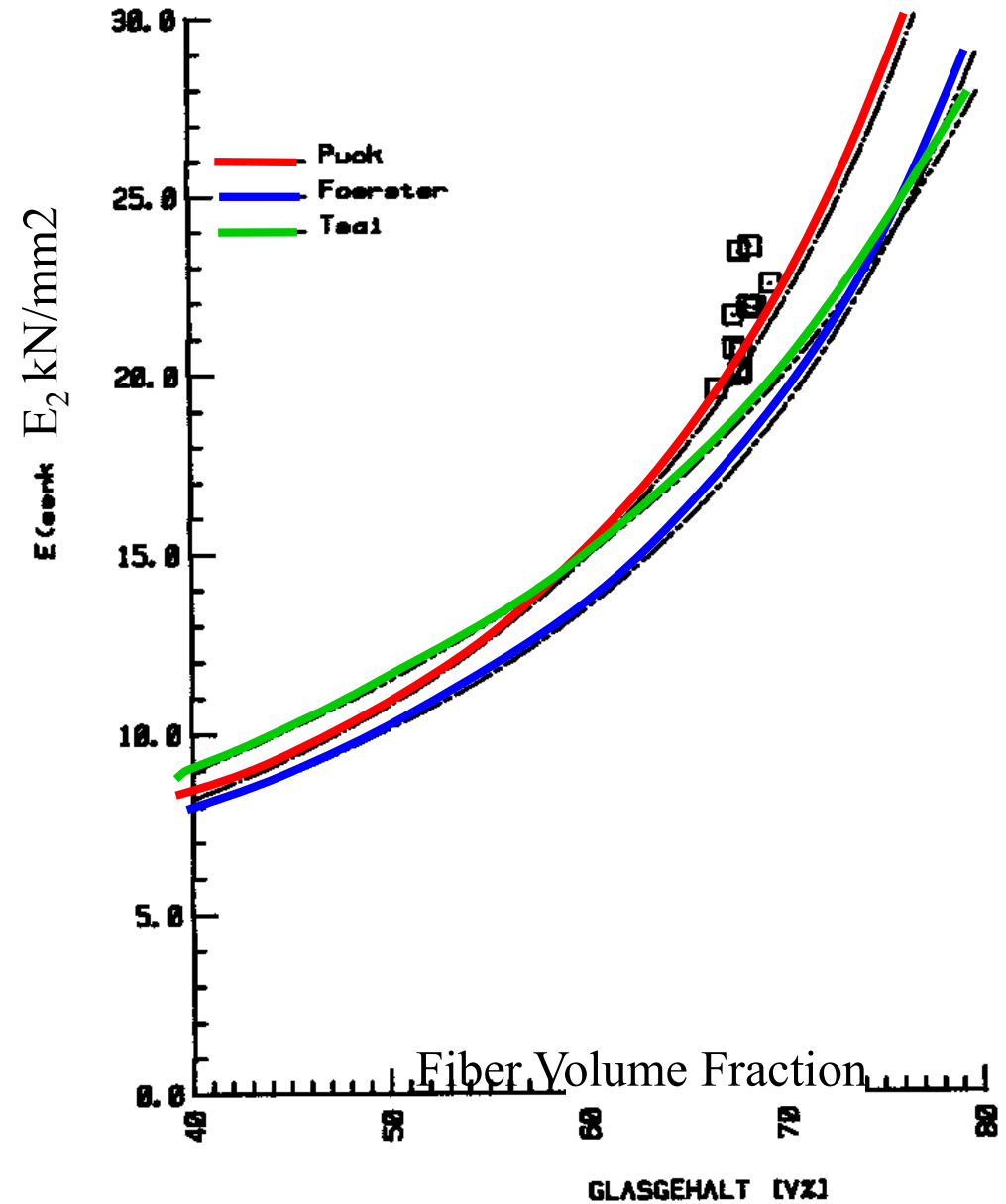
$$\eta = \frac{G_F / G_M - 1}{G_F / G_M + \xi} \quad \text{and} \quad \xi = 1$$

Comparisons for E_2

$$E_2 = E_M^o \frac{1 + 0.85\Phi_F^2}{\Phi_F E_M^o / E_F + (1 - \Phi_F)^{1.25}}$$

$$E_2 = E_M^o \frac{1}{\Phi_F E_M^o / E_F + (1 - \Phi_F)^{1.45}}$$

$$E_2 = \frac{E_M(1 + \xi \eta \Phi_F)}{1 - \eta \Phi_F}$$

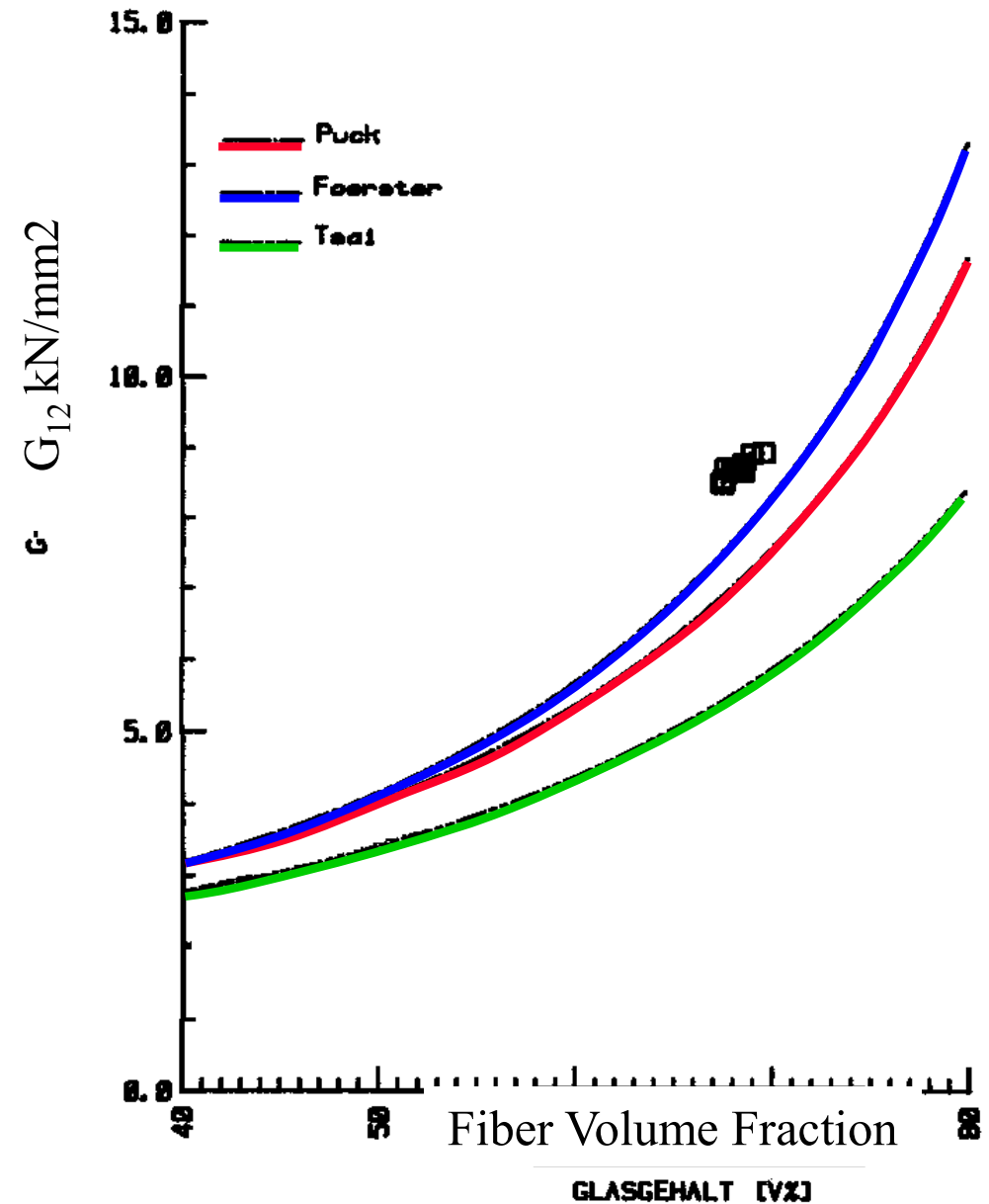


Comparisons for G_{12}

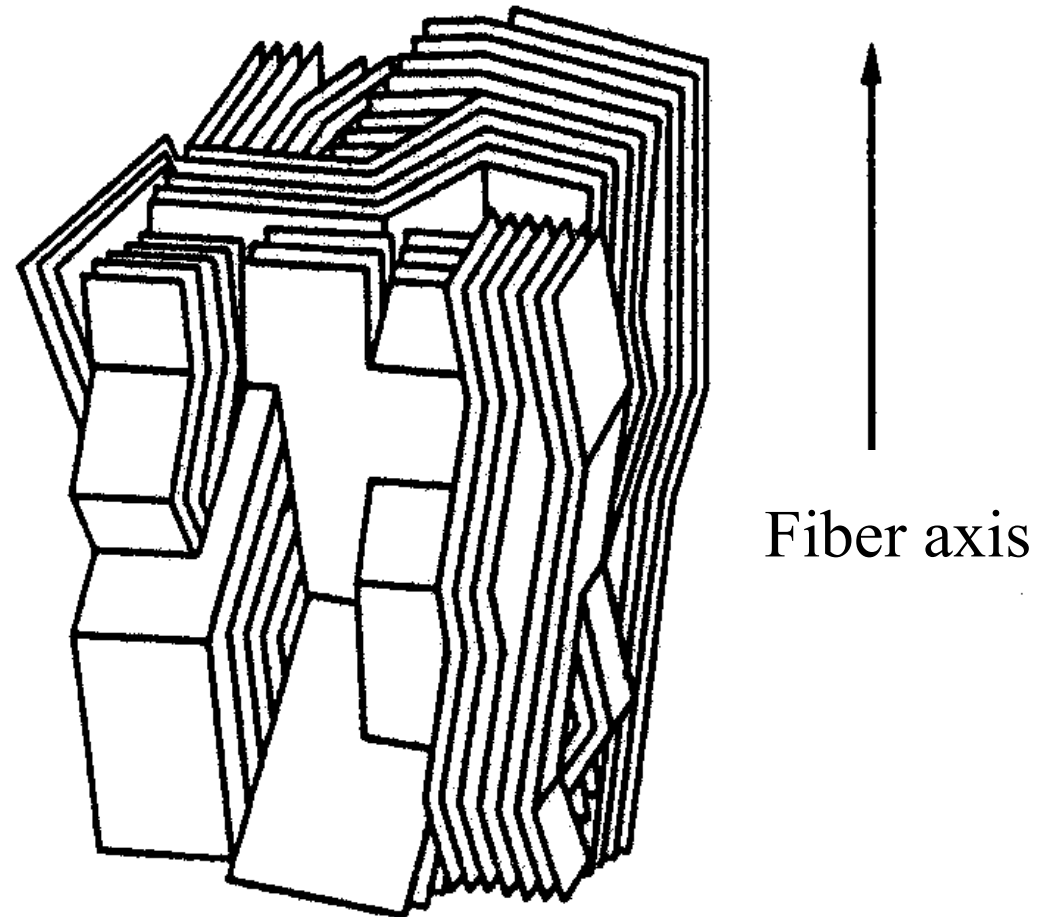
$$G_{12} = G_M \frac{(1 + 0.60\Phi_F^{0.5})}{\Phi_F G_M / G_F + (1 - \Phi_F)^{1.25}}$$

$$G_{12} = G_M \frac{(1 + 0.4\Phi_F^{0.5})}{\Phi_F G_M / G_F + (1 - \Phi_F)^{1.45}}$$

$$G_{12} = \frac{G_M (1 + \xi \eta \Phi_F)}{1 - \eta \Phi_F}$$



Anisotropic fibers



For anisotropic fibers like C-fibers following equations can be applied:

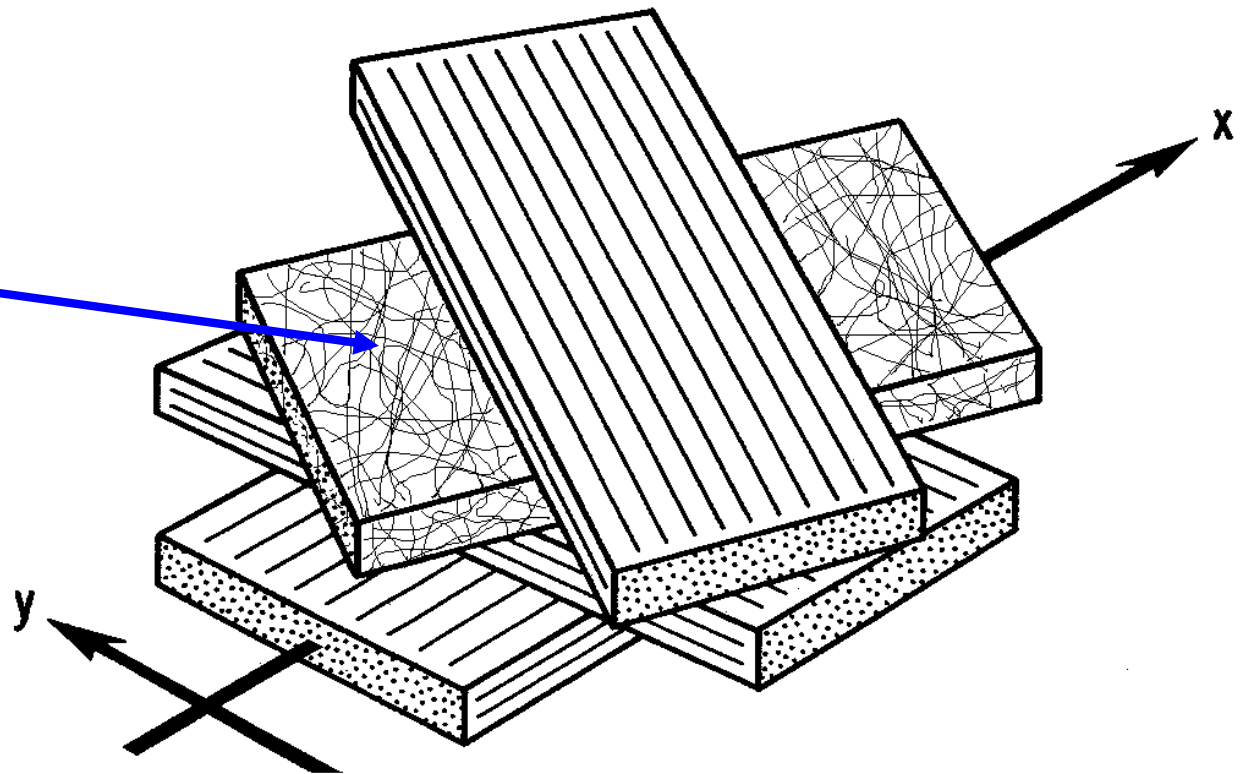
$$E_2 = \frac{E_M^o \left(1 + \Phi_F^3\right)}{\left(1 - \Phi_F\right)^{0.75} + 6 \Phi_F E_M^o / E_F}$$

$$E_M^o = \frac{E_M}{1 - \nu_M^2}$$

and

$$G_{12} = \frac{G_M \left(1 + 0.25 \Phi_F^{0.5} \right)}{\left(1 - \Phi_F \right)^{1.25} + 1.25 \Phi_F G_M / G_F}$$

Glass-Mat Lamina



Following equations can be applied to E-Glass Mat Lamina according to Puck :

$$E \approx 29'630 \quad \Phi_F - 4'710 \quad \Phi_F^2 + 3'920$$

$$\nu \approx 0.34 - 0.075 \quad \Phi_F$$

$$G \approx 10'970 \quad \Phi_F + 1'370$$

Elasticity constants of some UD-Laminas

Lamina type	T 300/5208	B (4)/5505	AS/3501	Scotchply 1002	Kevlar 49 / Epoxy
Fiber	C-Fibers from Toray	Boron	C –Fibers from Hercules	E-Glass	Aramid from E.I. Dupont de Nemours
Matrix	EP from Narmco	EP-Prepreg from Avco	EP-Prepreg from Hercules	EP-Prepreg from 3M	EP
Fiber volume fraction Φ (%)	70	50	66	45	60
Density (g/cm ³)	1.6	2.0	1.6	1.8	1.46
E_1 (N/mm ²)	181'000	204'000	138'000	38'600	76'000
E_2 (N/mm ²)	10'300	18'500	8'960	8'270	5'500
ν_{12}	0.28	0.23	0.30	0.26	0.34
G_{12} (N/mm ²)	7'170	5'590	7'100	4'140	2'300

...Elasticity constants of some UD-Laminas

Lamina type	T 300/5208	B (4)/5505	AS/3501	Scotchply 1002	Kevlar 49 / Epoxy
E_1 (N/mm ²)	181'000	204'000	138'000	38'600	76'000
E_2 (N/mm ²)	10'300	18'500	8'960	8'270	5'500
ν_{12}	0.28	0.23	0.30	0.26	0.34
G_{12} (N/mm ²)	7'170	5'590	7'100	4'140	2'300
S_{11} (mm ² /N)	$5.525 \cdot 10^{-6}$	$4.902 \cdot 10^{-6}$	$7.246 \cdot 10^{-6}$	$25.91 \cdot 10^{-6}$	$13.16 \cdot 10^{-6}$
S_{22} (mm ² /N)	$97.09 \cdot 10^{-6}$	$54.05 \cdot 10^{-6}$	$111.6 \cdot 10^{-6}$	$120.9 \cdot 10^{-6}$	$181.8 \cdot 10^{-6}$
S_{12} (mm ² /N)	$-1.547 \cdot 10^{-6}$	$-1.128 \cdot 10^{-6}$	$-2.174 \cdot 10^{-6}$	$-6.744 \cdot 10^{-6}$	$-4.474 \cdot 10^{-6}$
S_{33} (mm ² /N)	$139.5 \cdot 10^{-6}$	$172.7 \cdot 10^{-6}$	$140.8 \cdot 10^{-6}$	$241.5 \cdot 10^{-6}$	$434.8 \cdot 10^{-6}$
Q_{11} (N/mm ²)	181'800	205'000	138'000	39'160	76'640
Q_{22} (N/mm ²)	10'340	18'580	9'013	8'392	5'546
Q_{12} (N/mm ²)	2'897	4'275	2'704	2'182	1'886
Q_{33} (N/mm ²)	7'170	5'790	7'100	4'140	2'300

Thermal properties of a UD-Lamina:
Expansion coefficients (Book Geoff Eckold p59)

$$\alpha_1 = \alpha_F + \frac{\alpha_M - \alpha_F}{\frac{\Phi E_F}{(1 - \Phi)E_M} + 1}$$

And (Book Geoff Eckold p59)

$$\alpha_2 = \alpha_F \Phi + \alpha_M (1 - \Phi) + \nu_F \alpha_F \Phi + \nu_M \alpha_M (1 - \Phi) - [\nu_F \Phi + \nu_M (1 - \Phi)] \alpha_1$$

Failure Theories for a UD-Lamina

Failure Theories for a UD-Lamina

Following simple criteria can be applied to examine the **fiber failure**:

$$\left(\frac{\sigma_1}{\sigma_{1\max}} \right) = 1$$

$\sigma_{1\max}$ = Failure stress of a UD-Lamina in fiber direction

$$\sigma_{1\max} = \sigma_{F\max} \Phi + (1 - \Phi) \sigma_{M\max}$$

Matrix failure:

$$\left(\frac{\sigma_1}{\sigma_{1M\max}}\right)^2 + \frac{\sigma_2^2}{\sigma_{2C\max}\sigma_{2T\max}} + \frac{\sigma_{2C\max} - \sigma_{2T\max}}{\sigma_{2C\max}\sigma_{2T\max}}\sigma_2 + \left(\frac{\tau_{12}}{\tau_{12\max}}\right)^2 = 1$$

where

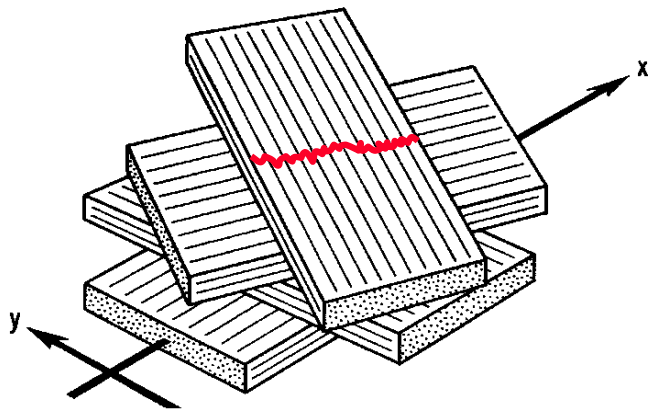
$$\sigma_{1M\max} = E_{1F}\varepsilon_{M\max}$$

$$\sigma_{2C\max} = \text{Compression strength perpendicular to fiber direction}$$

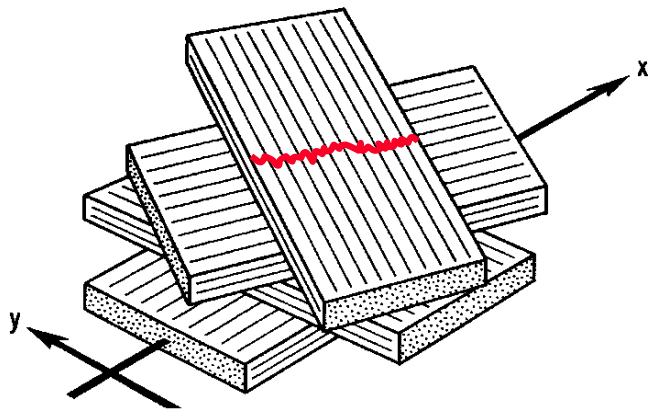
$$\sigma_{2T\max} = \text{Tensile strength perpendicular to fiber direction}$$

$$\tau_{12\max} = \text{Shear strength}$$

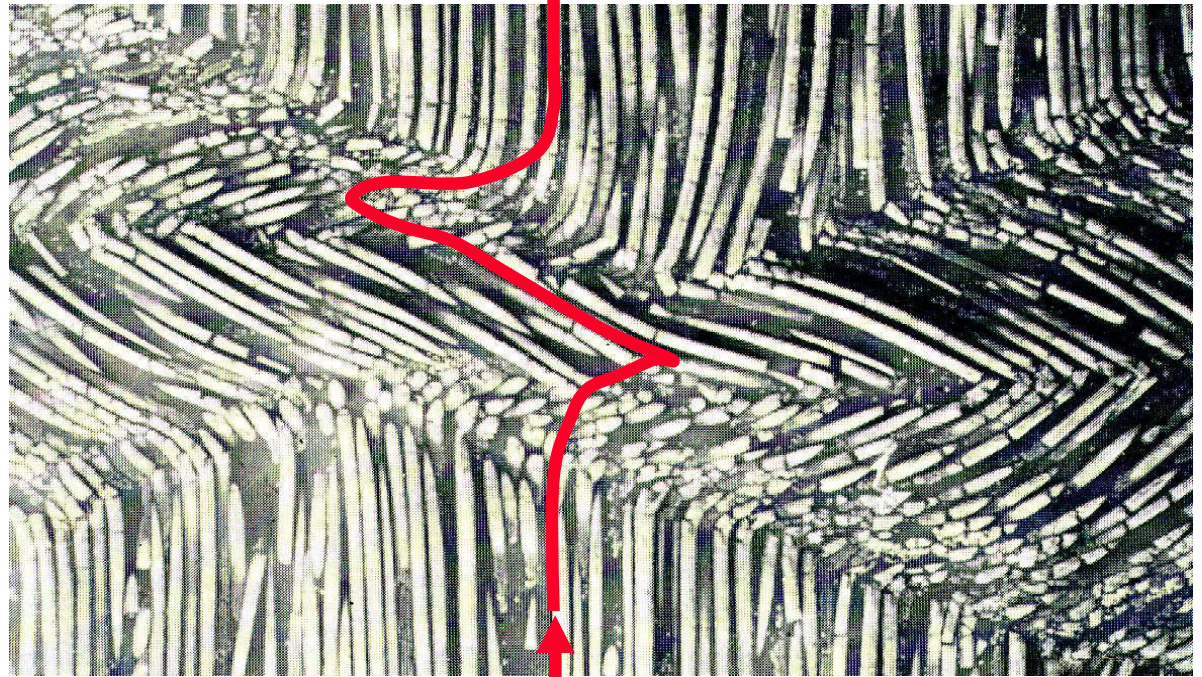
Fiber failure due to tensile stress



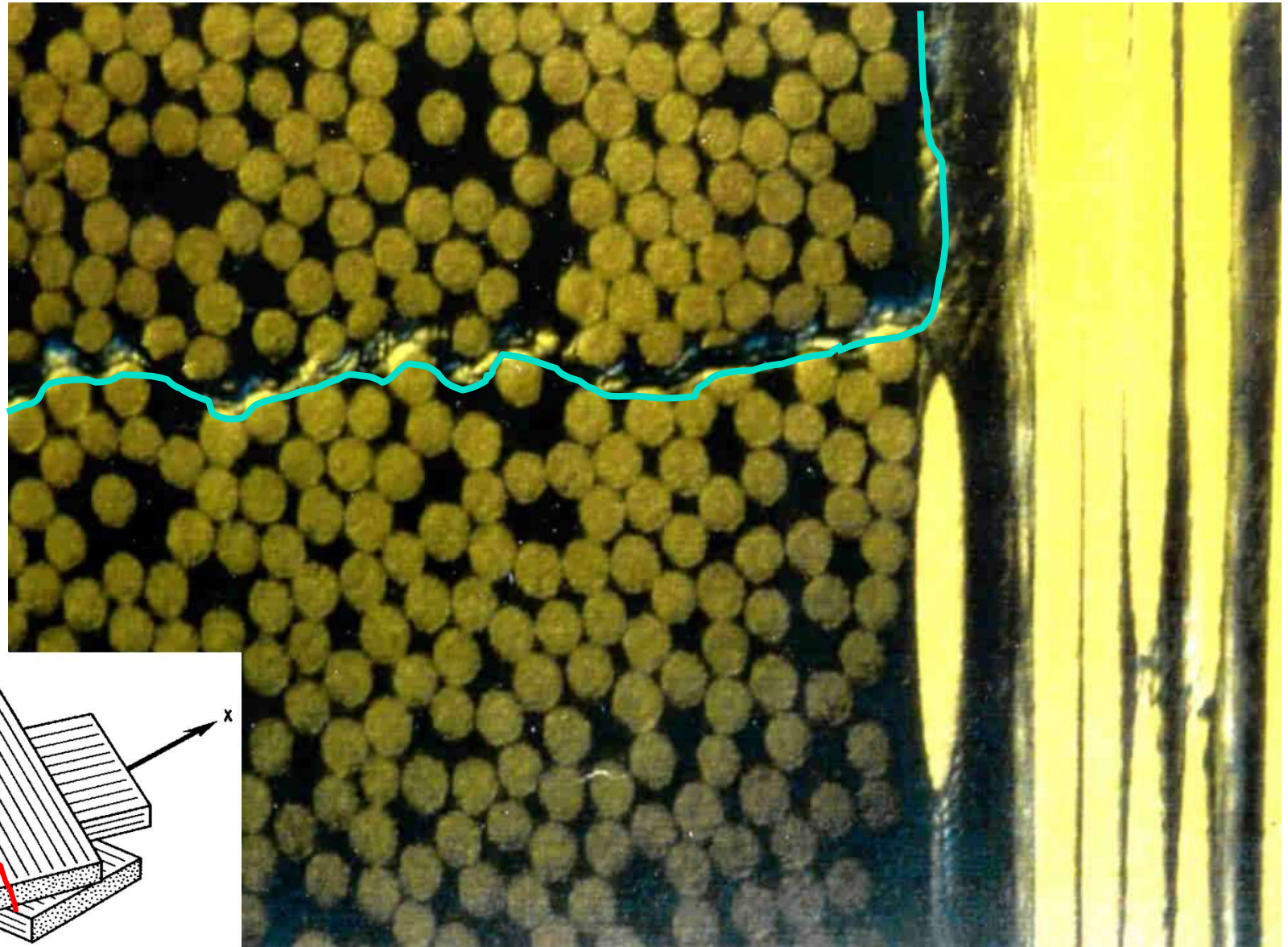
Fiber failure due to compression- stresses



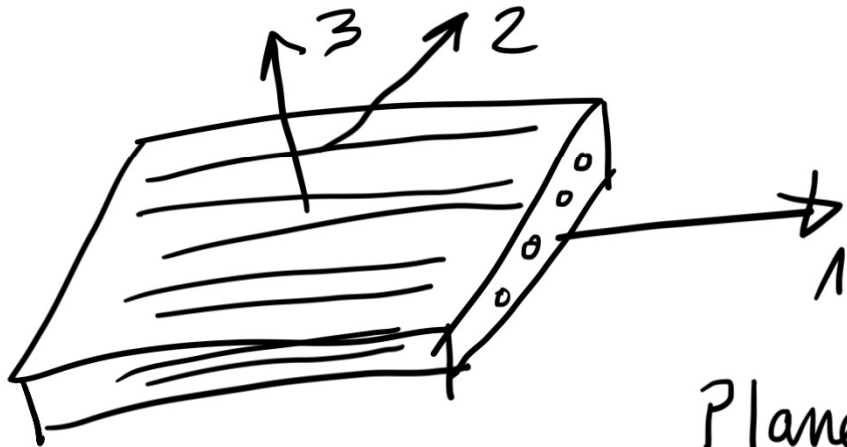
Fiber failure
due to
compression-
stresses



Matrix failure



UD-Lamina



orthotropic
9 independent
Const

$$E_{11}, E_{22}, E_{33}$$

$$G_{12}, G_{13}, G_{23}$$

$$\nu_{12}, \nu_{13}, \nu_{23}$$

Plane stress
4 ind. Const

$$E_{11}, E_{22}$$

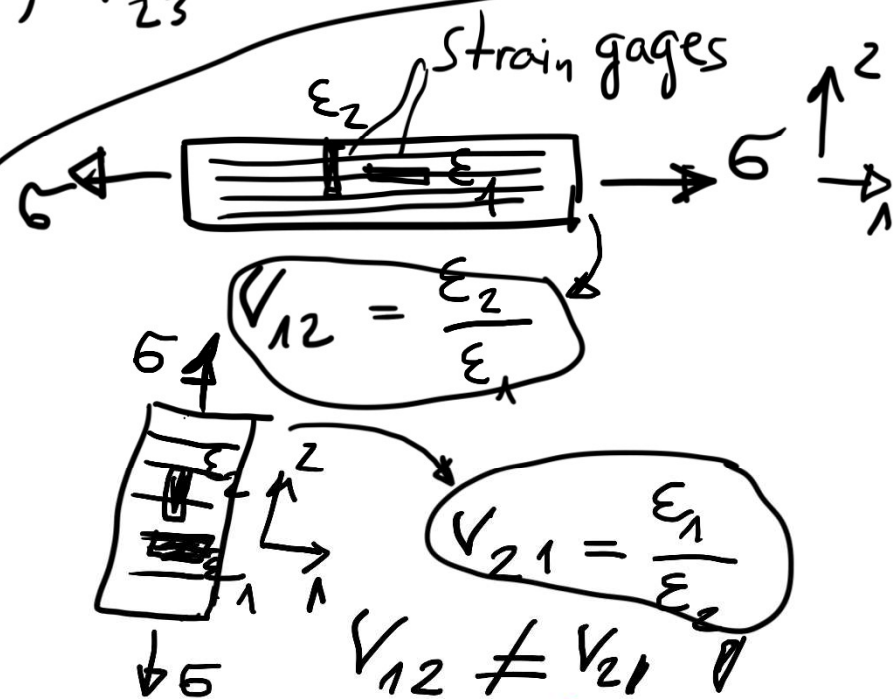
$$G_{12}, \nu_{12}$$

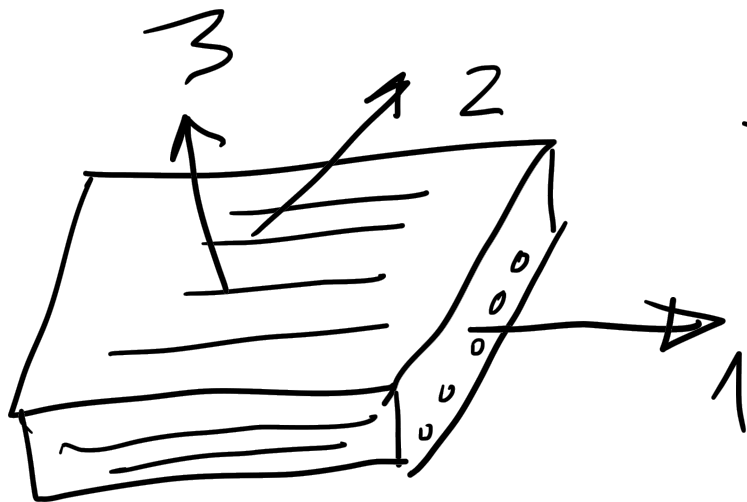
ν_{12} : Major Poisson ratio / ν_{21} : Minor

transversal isotropic
5 ind. Const.

$$E_{11}, E_{22} = E_{33}$$

$$G_{12}, \nu_{12}, G_{23}$$





UD - Lamina stiffness matrix

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

$$Q_{12} = Q_{21}!$$

$$\left. \begin{matrix} E_f, \nu_f \\ E_m, \nu_m \\ \phi_f \end{matrix} \right\} \rightarrow \begin{matrix} E_{11}, E_{22} \\ G_{12}, \nu_{12} \end{matrix} \rightarrow \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix}$$

Theoretical:

$$E_{11} = E_f \phi_f + E_m (1 - \phi_f)$$

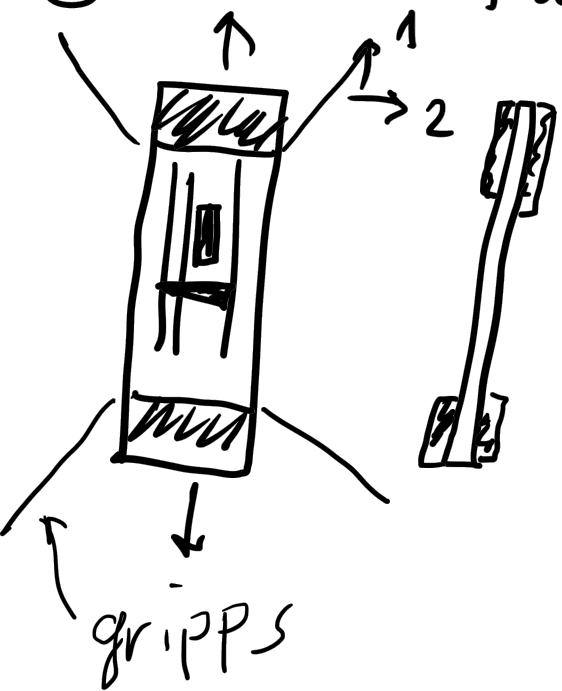
$$\nu_{12} = \nu_f \phi_f + \nu_m (1 - \phi_f)$$

$$\begin{matrix} E_{22} \\ G_{12} \end{matrix} \rightarrow \text{See semi empirical equations}$$

Experimental determination of UD-Lamina Properties

E_{11} , E_{22} , ν_{12} , G_{12}

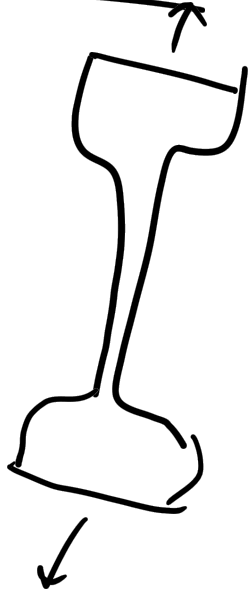
① uniaxial tensile test: E_{11} , ν_{12} , G_{11} failure



grips

$$\begin{pmatrix} \epsilon_{11} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 0 \end{pmatrix} \quad \text{or}$$

Measurement uncertainties:



Ahmadreza Hejazi

Subject: _____ Day: _____ Time: _____
 Date: _____

①
$$\begin{bmatrix} \sigma_{11} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 0 \end{bmatrix}$$

$\sigma_1, \epsilon_{11}, \epsilon_{22}$ Known

$$\nu_{12} = -\frac{\epsilon_{22}}{\epsilon_{11}} \quad \nu_{21} = \nu_{12} \frac{E_2}{E_1}$$

①
$$\sigma_{11} = \frac{E_1}{1 - \nu_{21}\nu_{12}} \epsilon_{11} + \frac{\nu_{12} E_2}{1 - \nu_{21}\nu_{12}} \epsilon_{22}$$

$$\sigma_1 = \frac{E_1}{1 - \nu_{12}^2 \frac{E_2}{E_1}} \epsilon_{11} + \frac{\nu_{12} E_2}{1 - \nu_{12}^2 \frac{E_2}{E_1}} (-\nu_{12} \epsilon_{11})$$

$$\sigma_1 = E_1 \epsilon_{11} \left(\frac{E_1 - \nu_{12}^2 E_2}{E_1 - \nu_{12}^2 E_2} \right) \rightarrow \boxed{\sigma_1 = E_1 \epsilon_{11}}$$

②
$$\frac{\nu_{21} E_1}{1 - \nu_{21}\nu_{12}} \epsilon_{11} + \frac{E_2}{1 - \nu_{21}\nu_{12}} \epsilon_{22} = 0$$

$$(\nu_{12} E_2) \epsilon_{11} + E_2 (-\nu_{12} \epsilon_{11}) = 0$$

$$0 = 0 \quad \checkmark$$

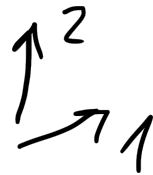
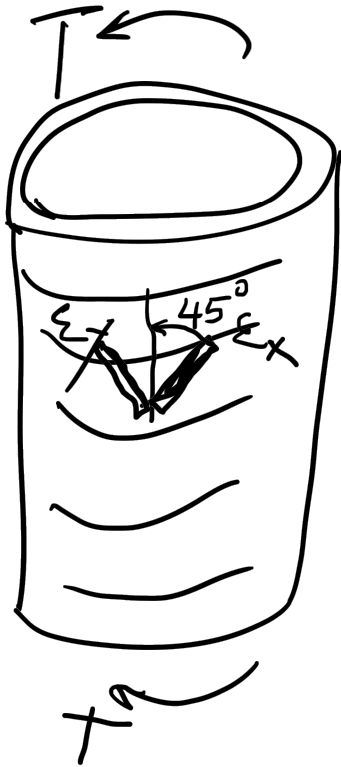
② uniaxial tensile test : E_{22} , ν_{21} , ϵ_{22} failure



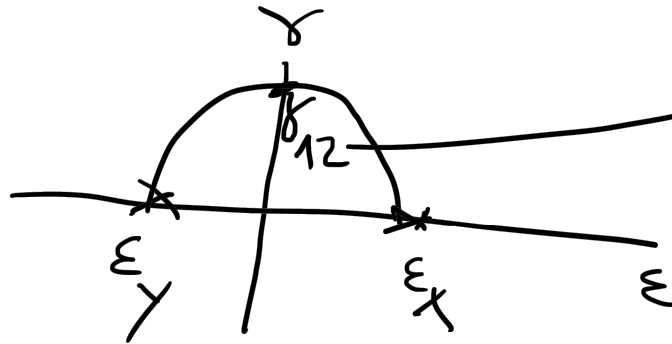
better



③ Torsion test : G_{12} , τ_{12} failure

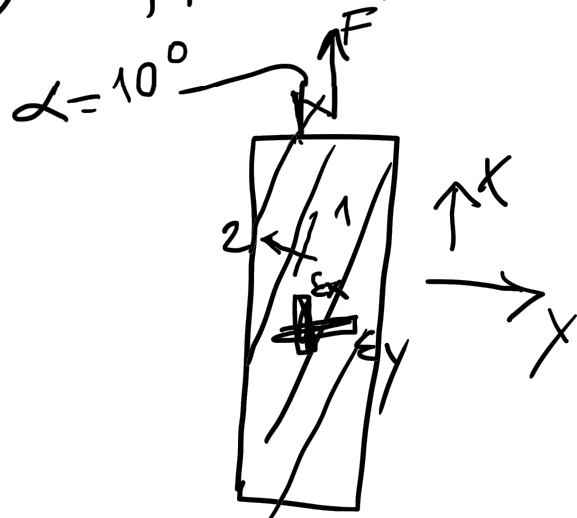


$$\tau_{12} = G_{12} \cdot \gamma_{12}$$



$$\tau_{12} = f(T, R, t)$$

④ off-axis tensile test : G_{12}



$$\begin{pmatrix} \sigma_x \\ 0 \\ 0 \end{pmatrix}$$

rotate $\rightarrow$

$$\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix}$$

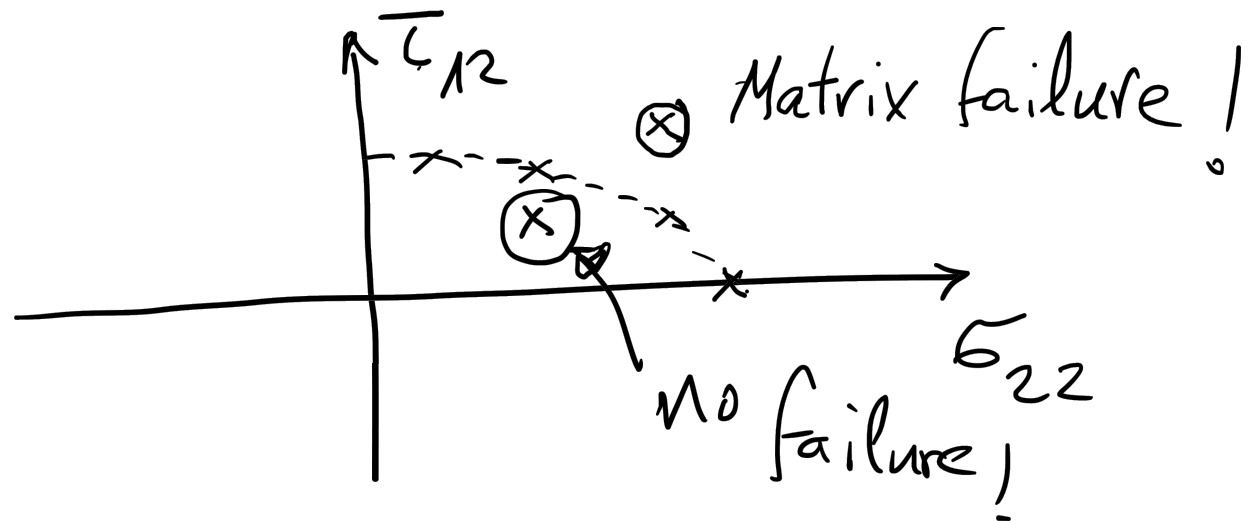
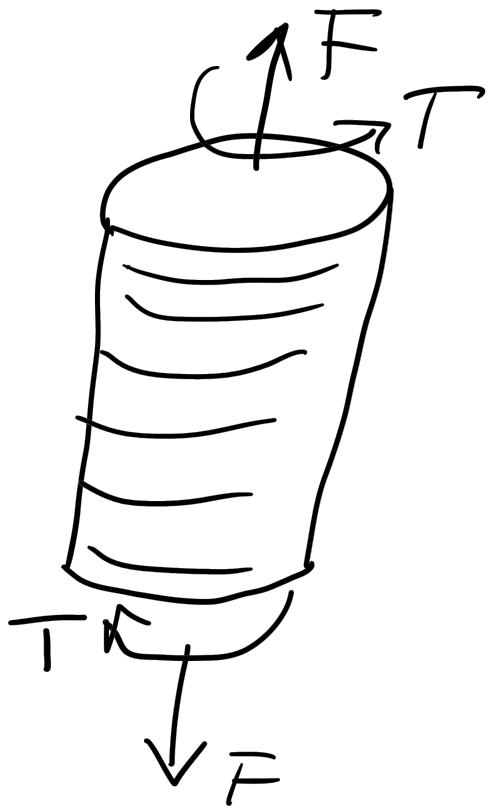
$$\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ 0 \end{pmatrix}$$

rotate $\rightarrow$

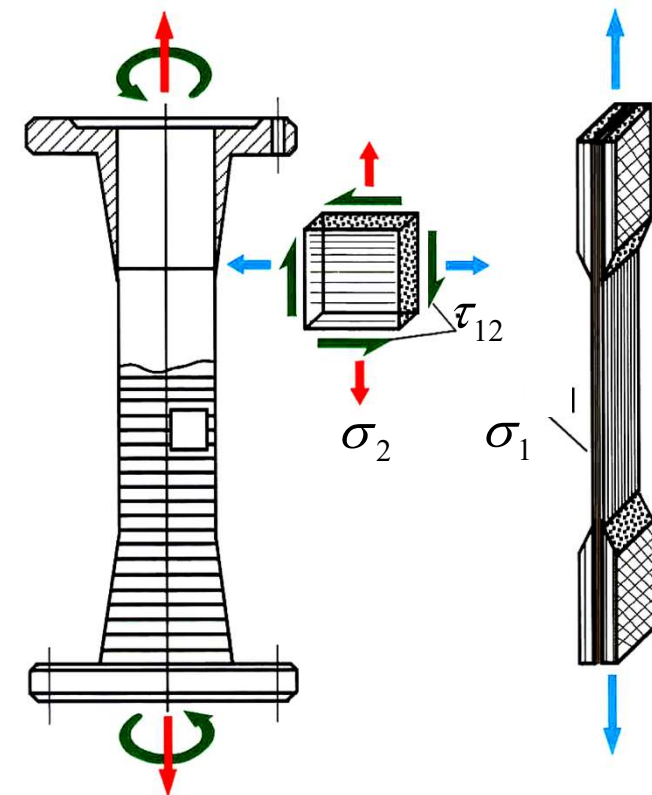
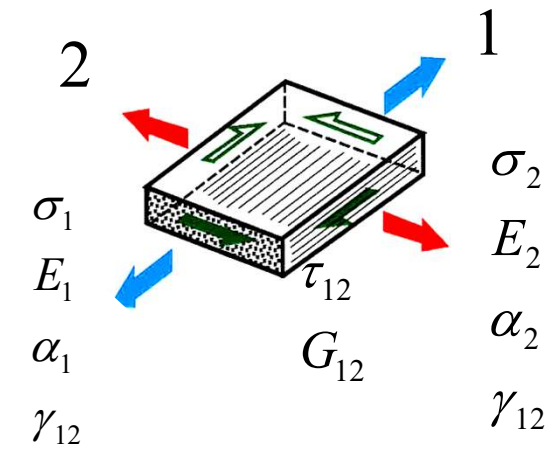
$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{pmatrix} \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{pmatrix} \Rightarrow \gamma_{12} = S_{66} \tau_{12} \rightarrow G_{12}$$

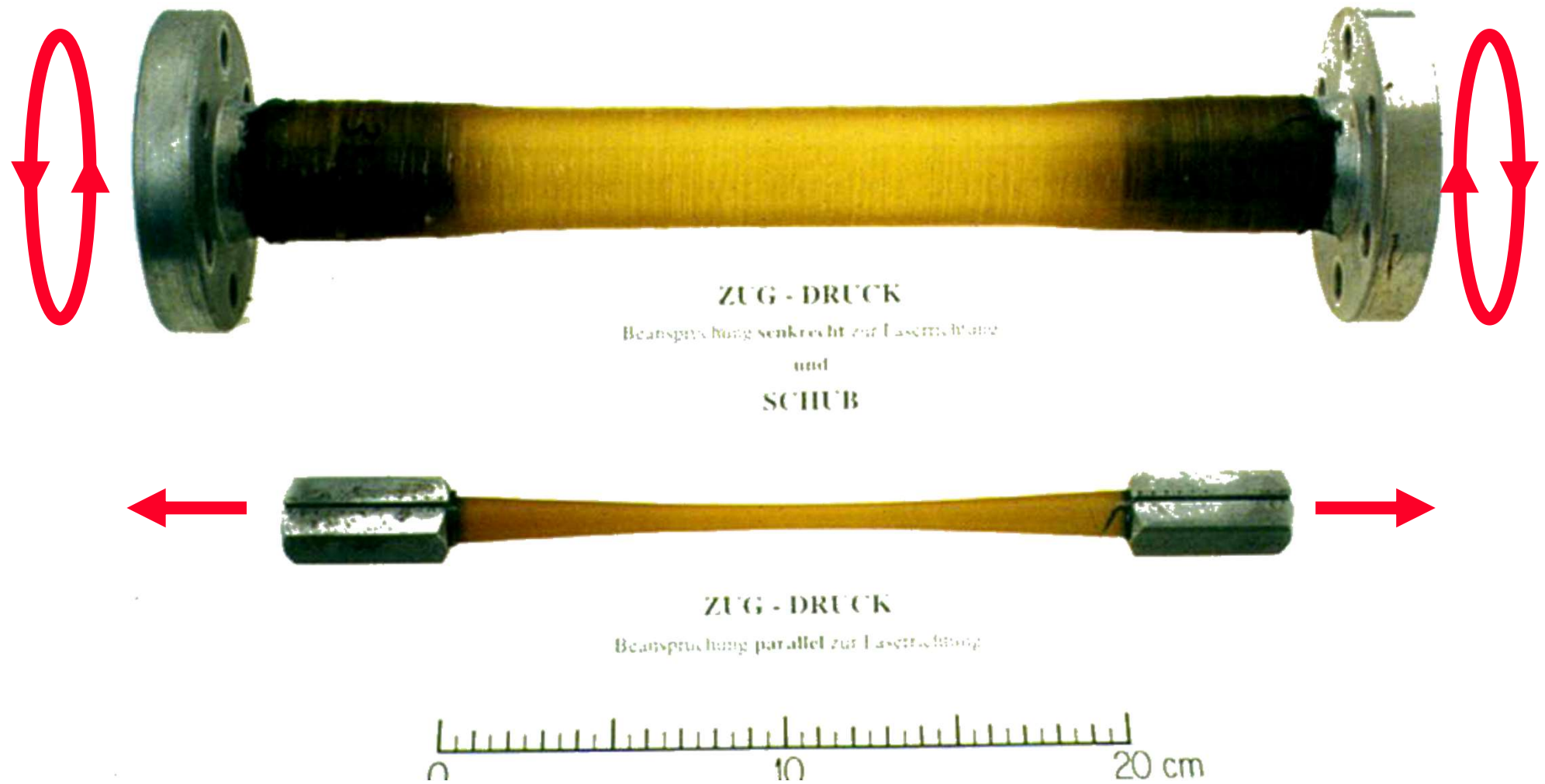
⑤ Biaxial tension/torsion test



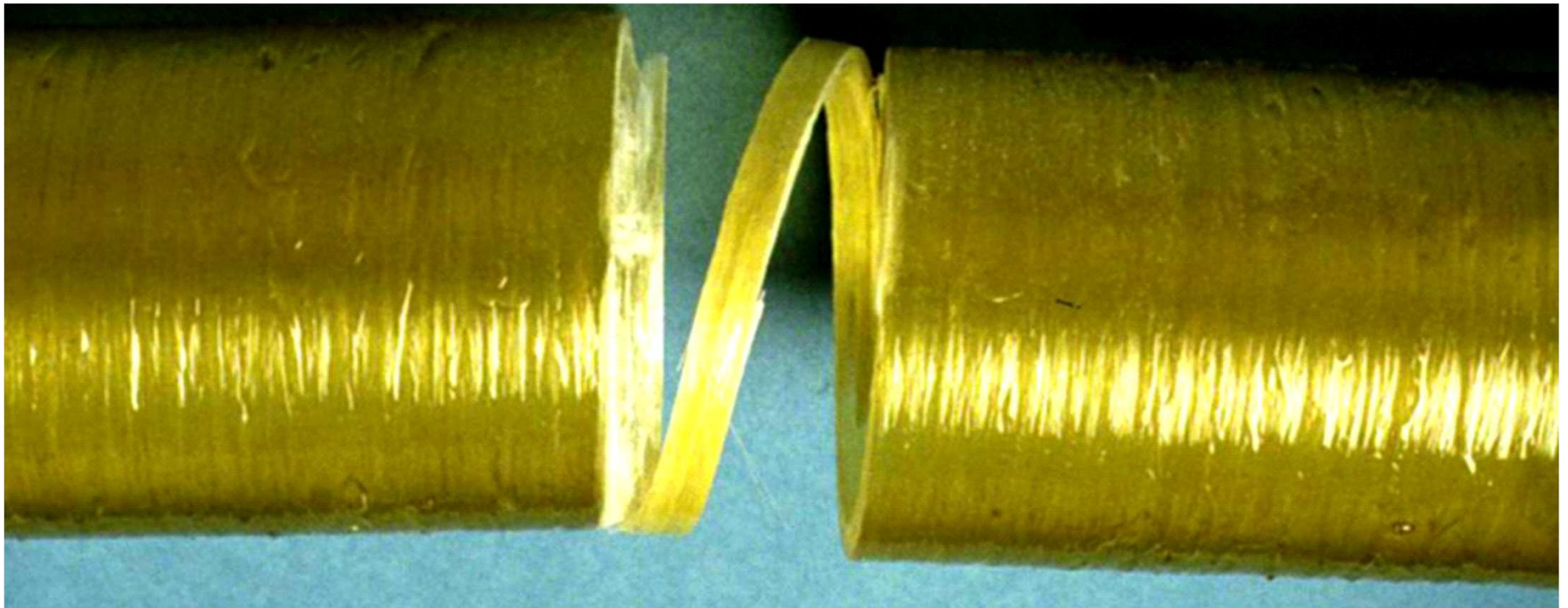
Experimental determination
of the UD-Lamina
properties:



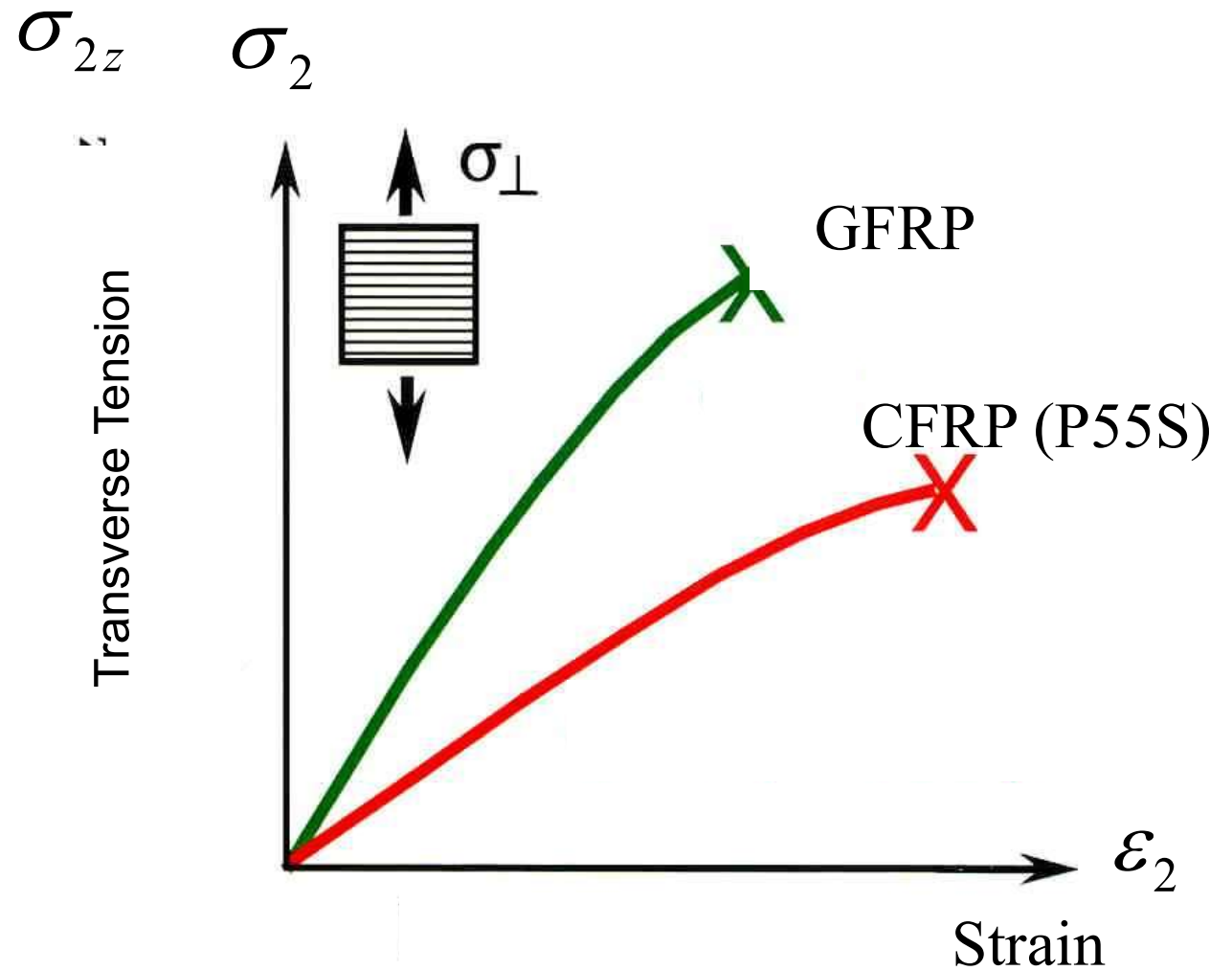
Torsion and Tensile Samples



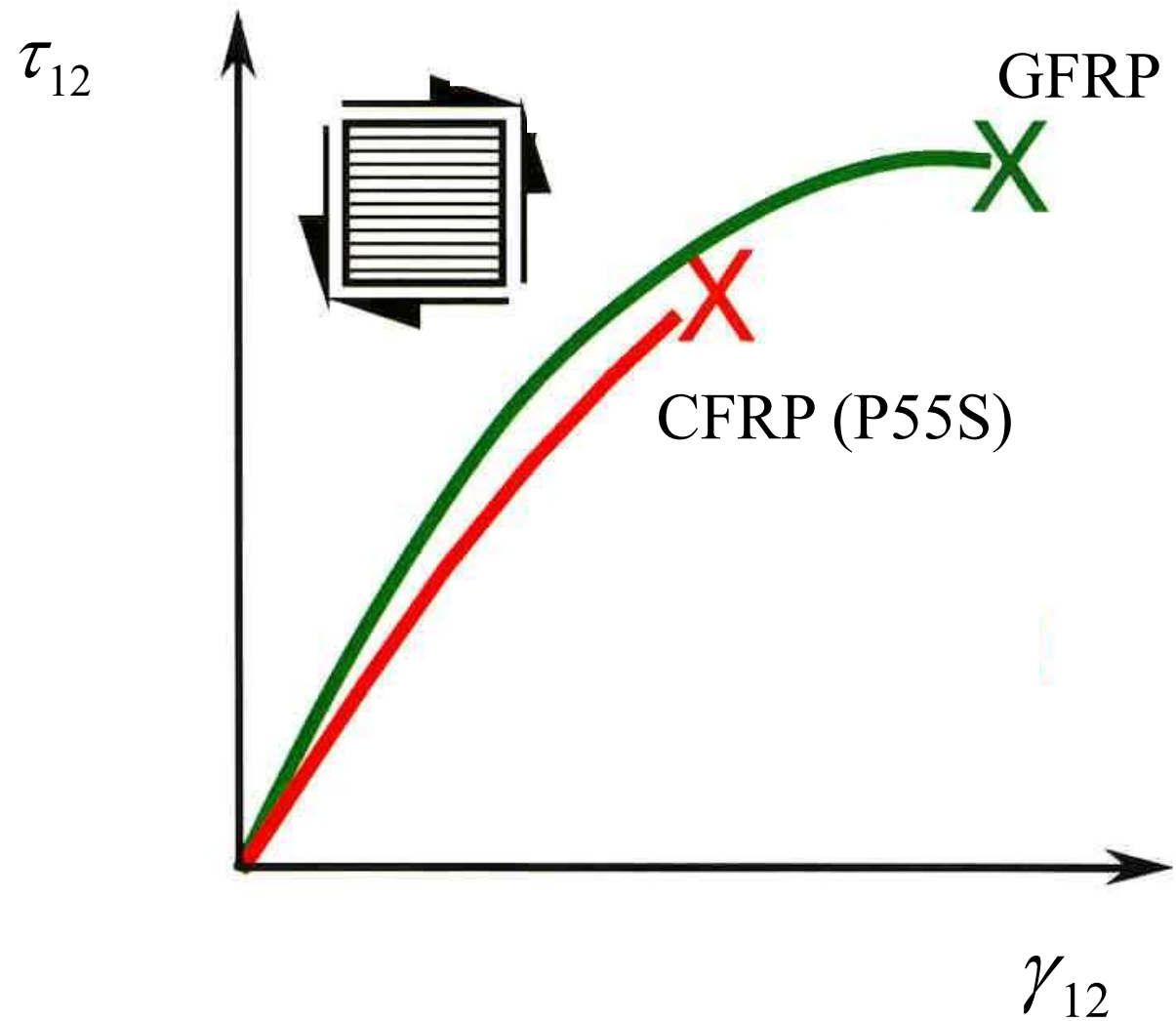
A torsion sample after the test



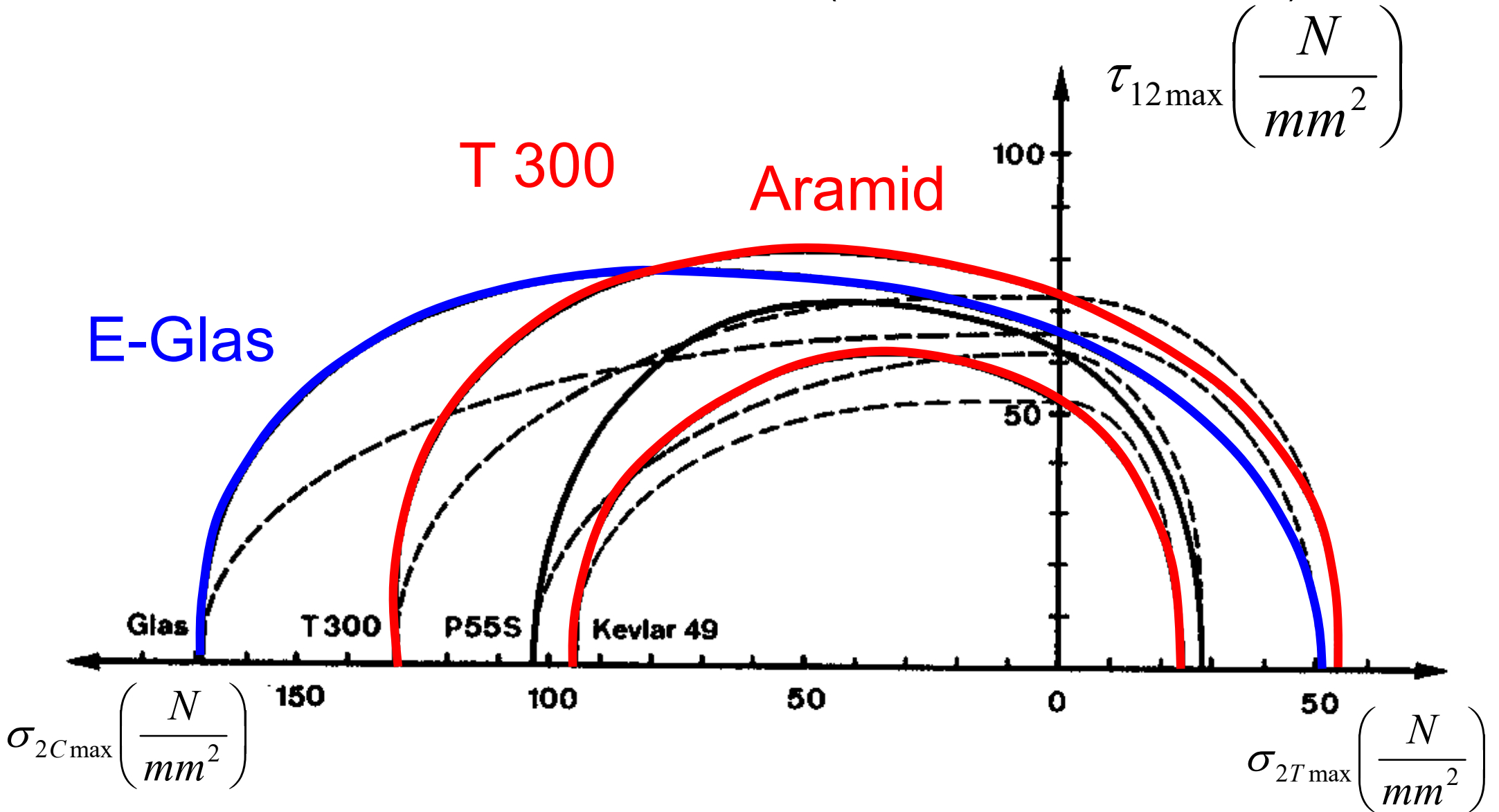
Transverse Tension



Shear (torsion test)



Combined shear and transverse stresses (biaxial tension-torsion test)



Strength of some UD-Laminas

Lamina type	T 300/5208	B (4)/5505	AS/3501	Scotchply 1002	Kevlar 49 / Epoxy
$\sigma_{1Tmax}(N/mm^2)$	1500	1260	1447	1062	1400
$\sigma_{1Cmax}(N/mm^2)$	1500	2500	1447	610	235
$\sigma_{2Tmax}(N/mm^2)$	40	61	51.7	31	12
$\sigma_{2Cmax}(N/mm^2)$	246	202	206	118	53
$\tau_{12max}(N/mm^2)$	68	67	93	72	34

< Mechanics of a Lamina >

List of Symbols :

- $\epsilon_{11}, \epsilon_{22}, \dots, \gamma_{23}, \gamma_{13}, \dots$: normal and shear strains
 UD-Lamina; Local directions
- $\sigma_{11}, \sigma_{22}, \dots, \tau_{23}, \tau_{13}, \dots$: normal and shear stresses
 UD-Lamina; Local directions
- $S_{11} = \frac{1}{E_{11}}; S_{12} = -\frac{\nu_{12}}{E_{11}}, \dots$: Components of Compliance matrix
 $E_{11}, E_{22}, \dots, G_{12}, \dots$ E- and G-modulus
 in principal direction
- $C_{11} = \frac{S_{22}S_{33} - S_{23}S_{23}}{S}; C_{12}, \dots$: Components of stiffness matrix
- $\nu_{12}, \nu_{13}, \nu_{23}$: poisson's ratios
- $\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$: stiffness matrix of a UD-Lamina
- $\begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix}$: Compliance matrix of a UD-Lamina
- E_F, ν_F : fibre E-modulus and fibre Poisson's ratio
- E_M, ν_M : Matrix E-modulus and Matrix Poisson's ratio
- ϕ (or ϕ_f or V^f) : fibre volume fraction
- $E_1 = E_{11}; E_2 = E_{22}$: UD - Elastic moduli
- $\alpha_1; \alpha_2$: UD Thermal expansion coefficients
 in local direction
- $\alpha_x; \alpha_y, \alpha_{xy}$: UD Thermal expansion coefficients
 in global direction