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## **Mechanics of a Lamina**

Book Geoff Eckold, Chapter 3, pp 49-65

A **Laminate** is consisting of several **Laminas or Plies or Layers.**

A **Lamina** is consisting of Fibers and Matrix.

Micromechanics (in  $\mu\text{m}$ -mm range) is dealing for example with the determination of **Lamina** constitutive properties from those of Fiber and Matrix, Fiber-Matrix interface stresses, etc.

## **Assumptions:**

- Linear Elasticity: Matrix and Fiber behave as linear elastic material (viscoelasticity of Matrix: see previous chapter)
- Perfect bond, no strain discontinuity across interface
- Fibers are arranged in a regular or repeating array

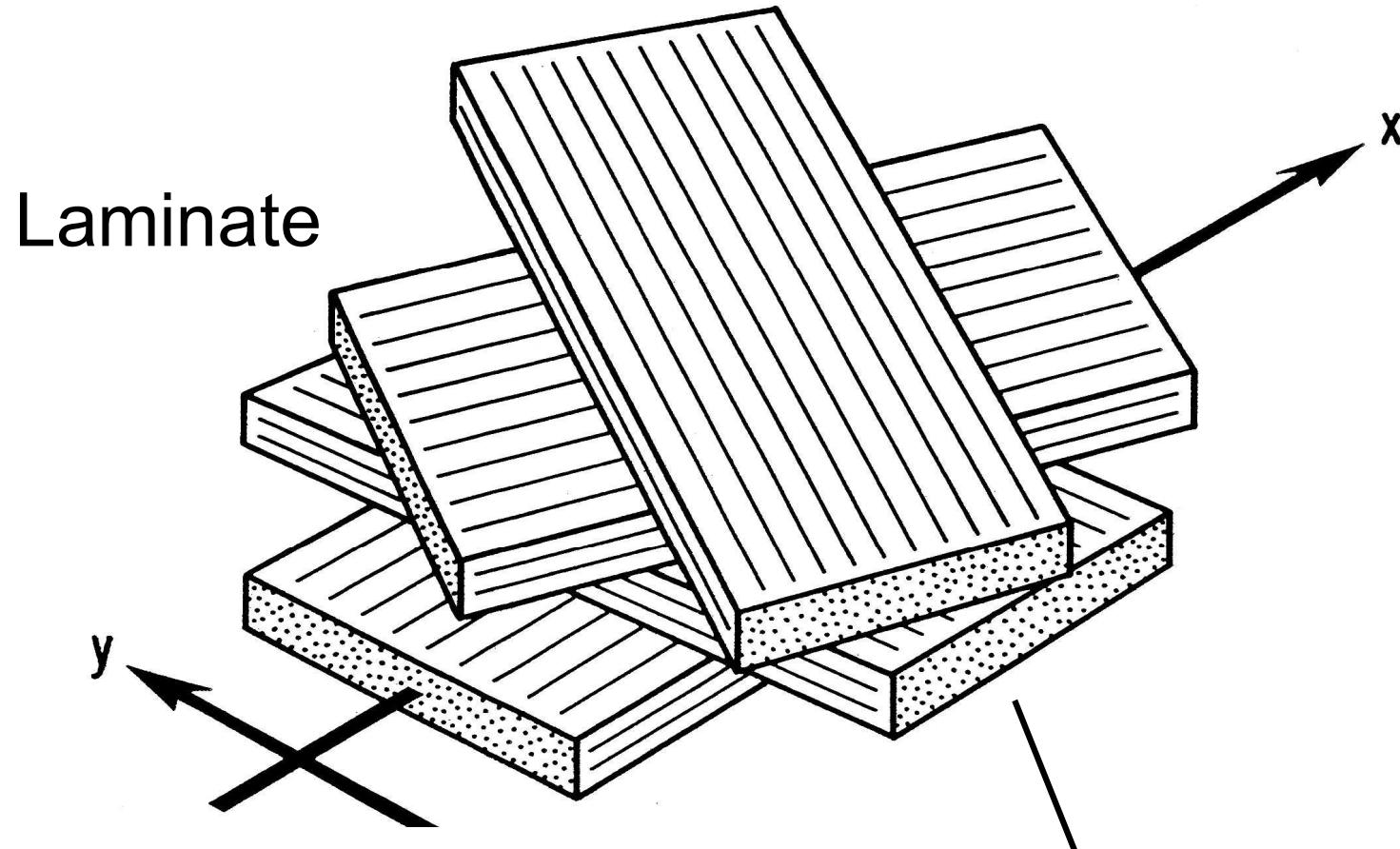
## **Functional requirements for Fibers:**

- High E-Modulus
- High ultimate strength
- Low variation between individual fibers
- Retain the strength during handling and fabrication
- Uniform diameter and surface

## **Functional requirements for Matrix:**

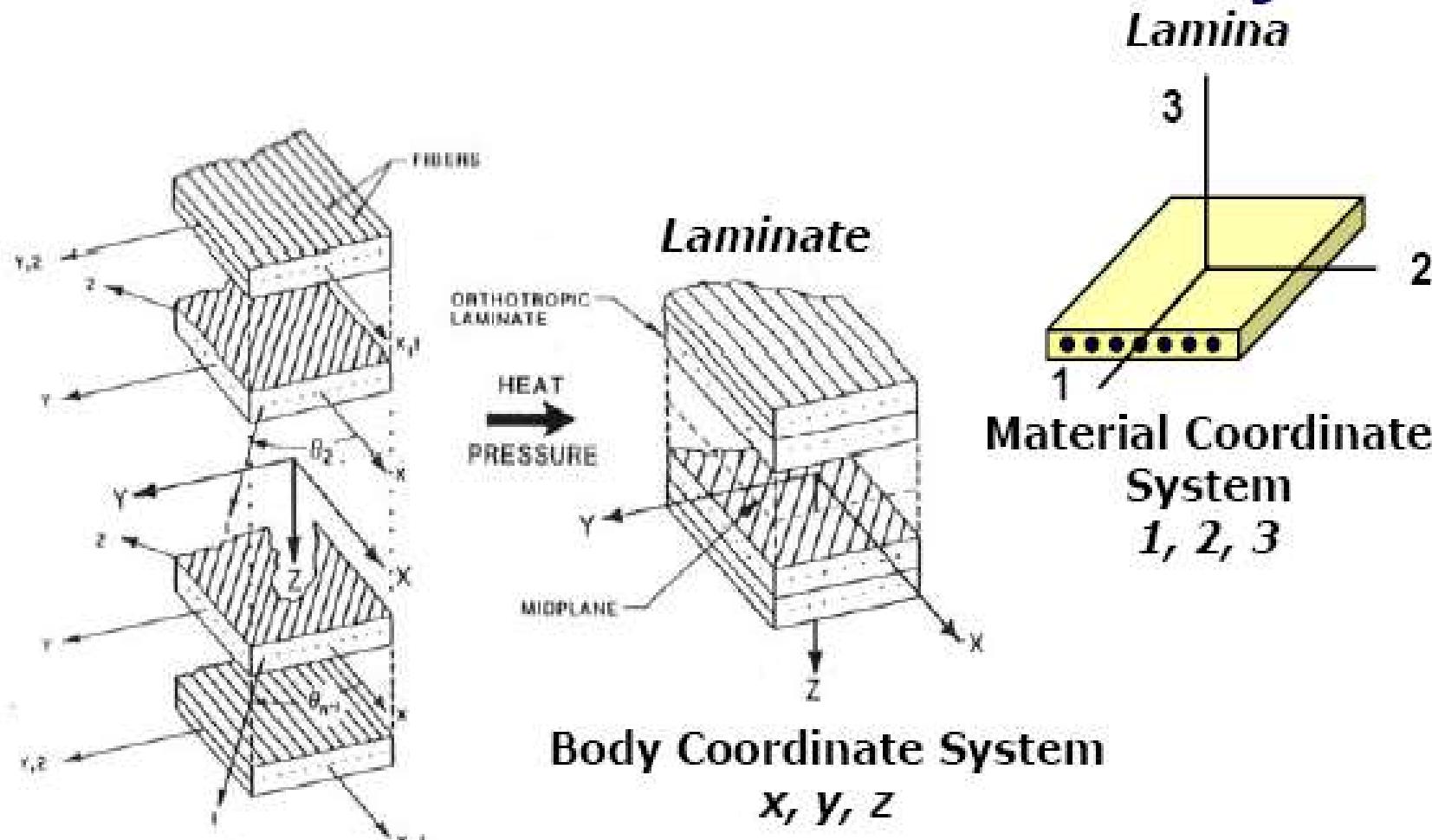
- Bind together the fibers and protect their surfaces
- Transfer stresses to the fibers efficiently
- Chemically compatible with fibers over a long period
- Thermally compatible with fibers

## Modell of a Laminate



A Unidirectionally Reinforced Lamina  
(UD)

# Lamina – Laminate Coord. Sys.



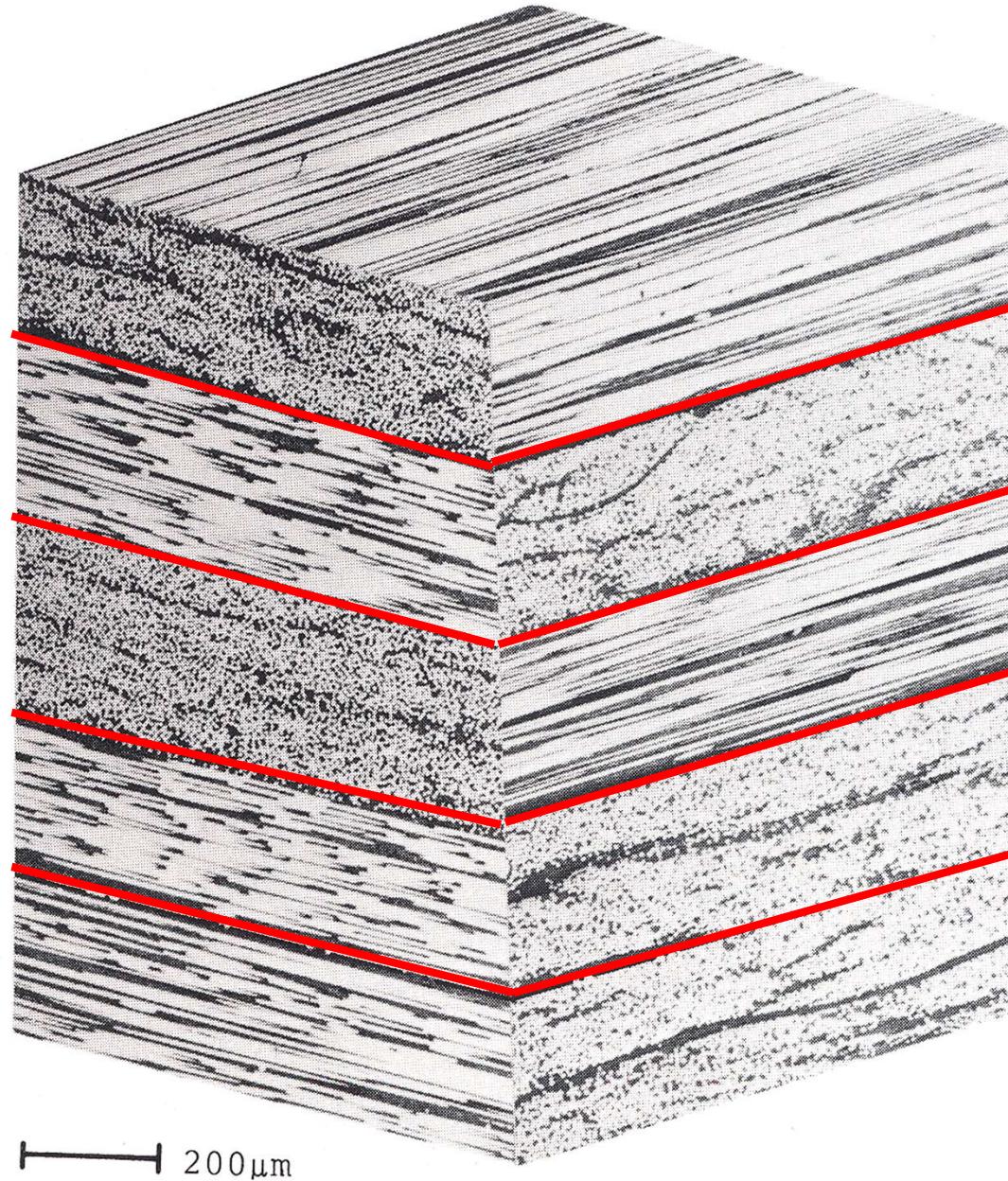
**A Laminate**

**UD-Laminas**



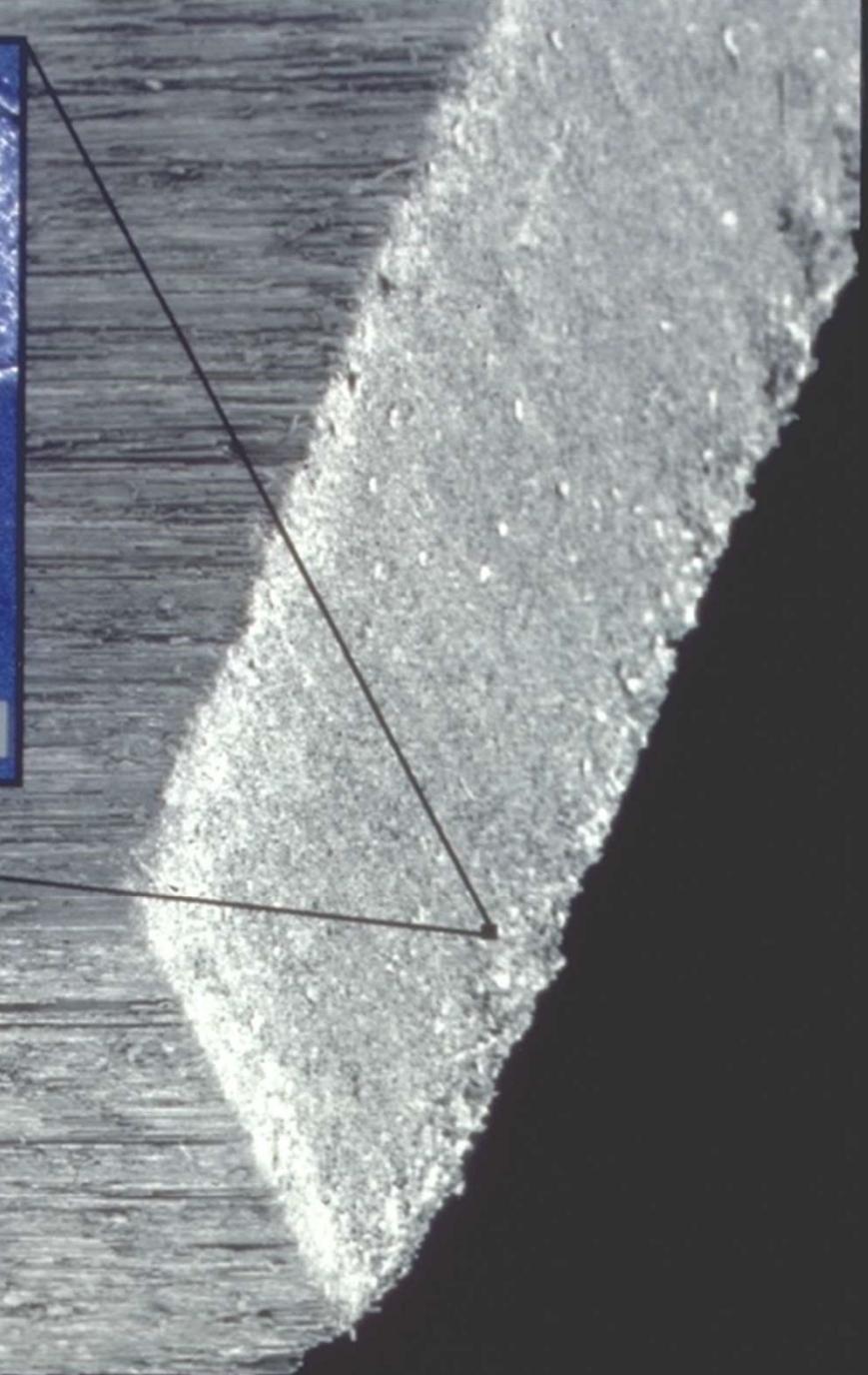
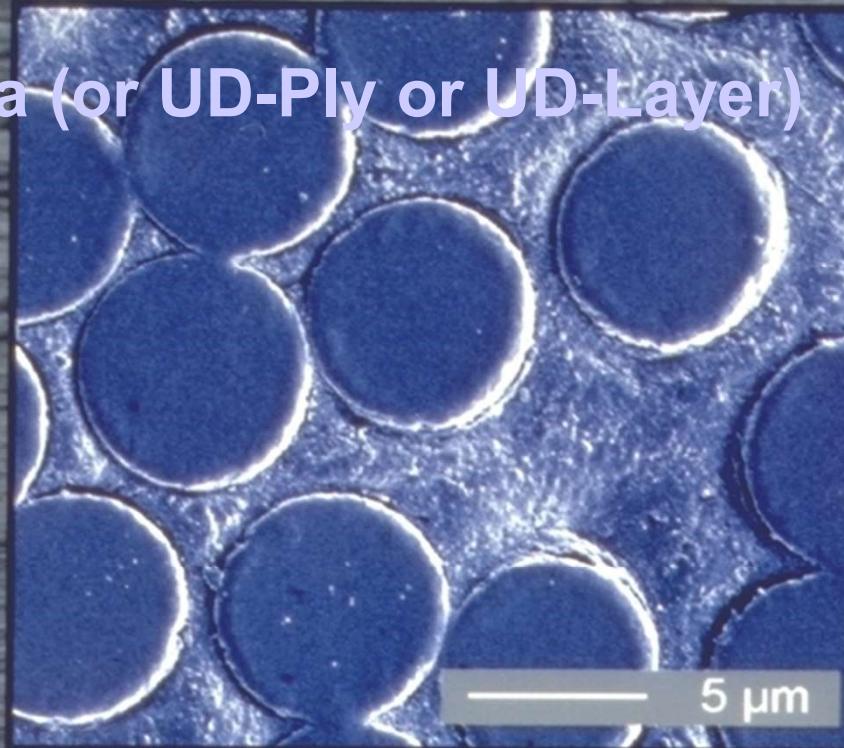
200  $\mu\text{m}$

# A Laminate



200  $\mu\text{m}$

**UD-Lamina (or UD-Ply or UD-Layer)**



— 300 μm

## Definitions:

- **Homogeneous**: Properties are not function of the position of the material points

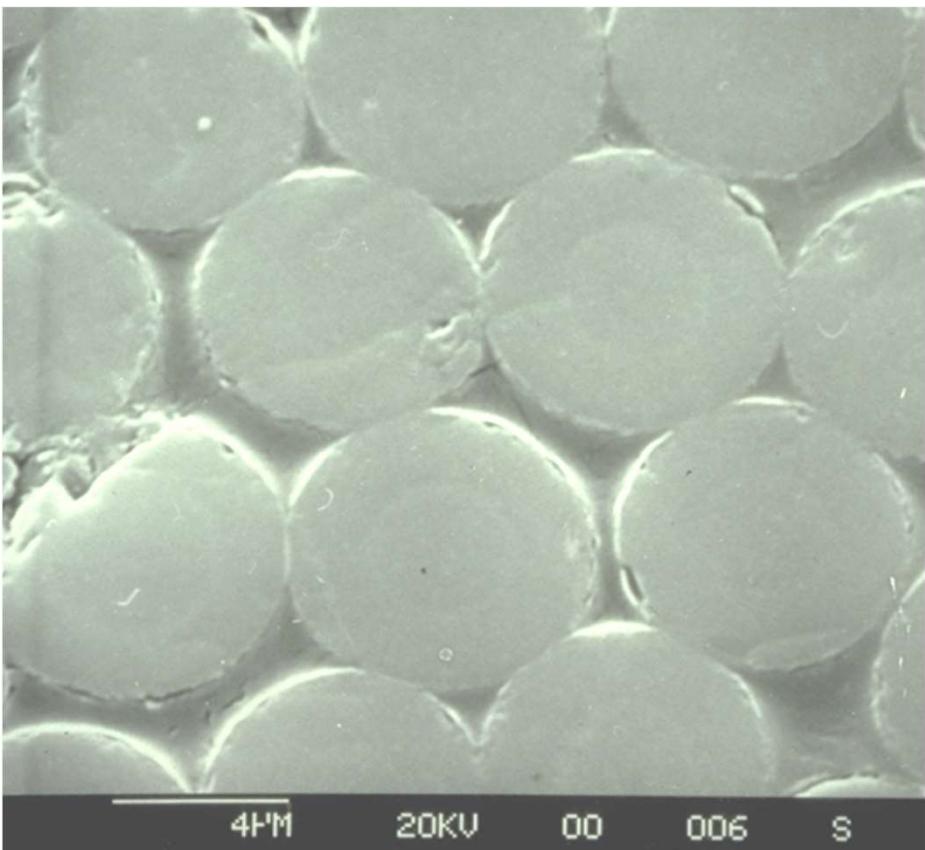
- **Isotropy**: Properties are not function of the orientation.

2 independent material constants: E and  $\nu$

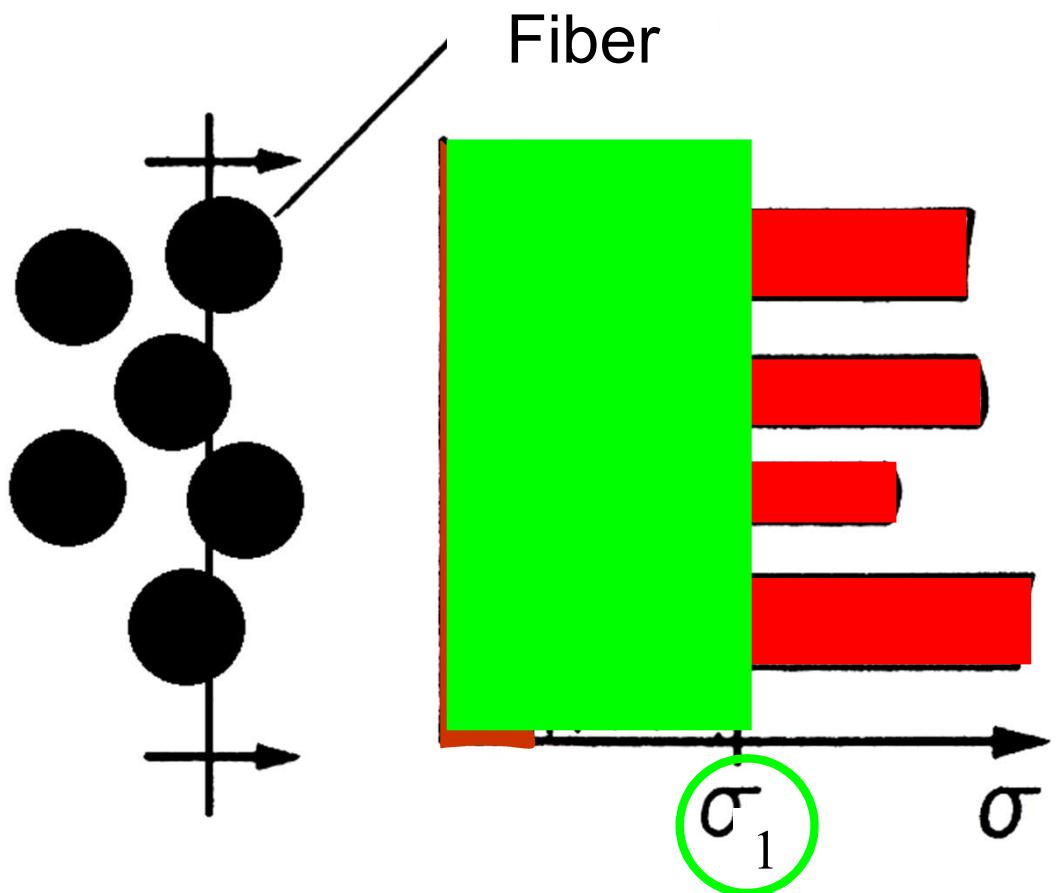
- **Anisotropy**: Properties are function of the orientation with no planes of symmetry.

21 independent material constants

# Micromechanics

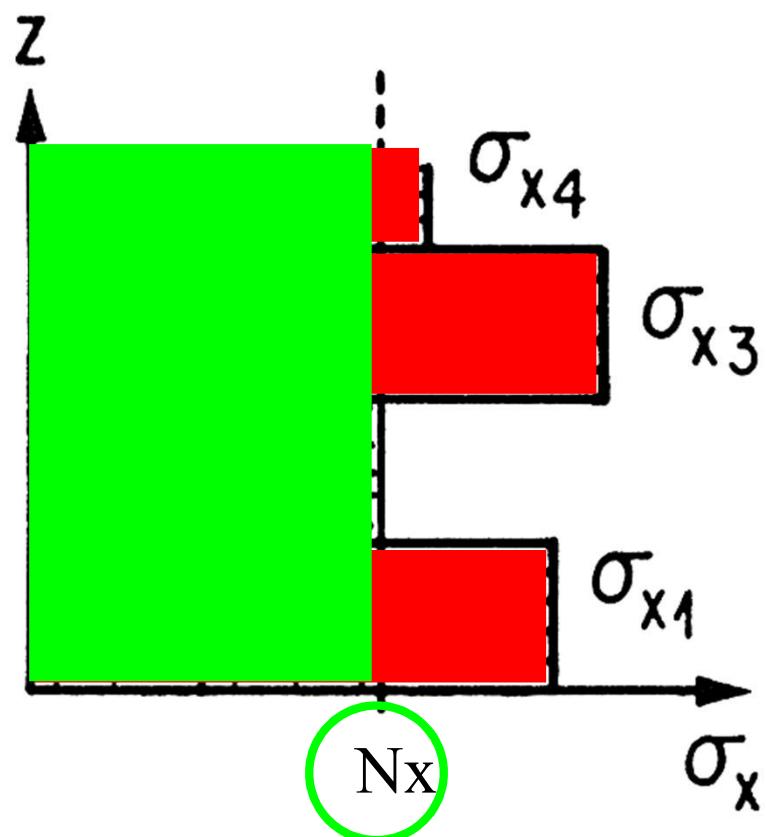


a)



## Macromechanics

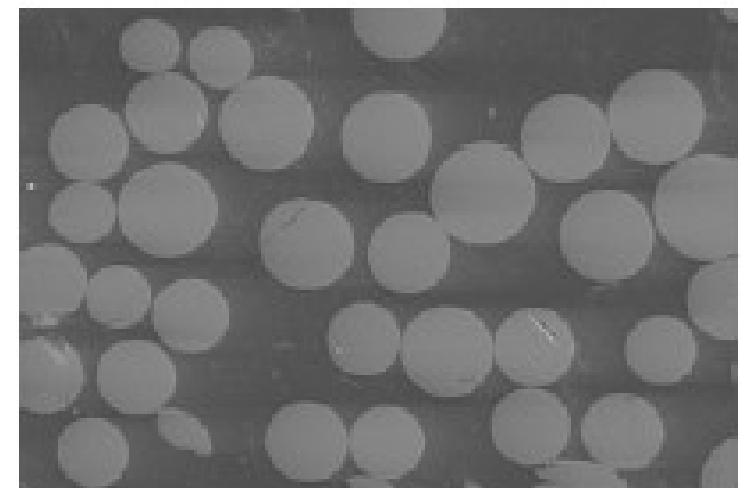
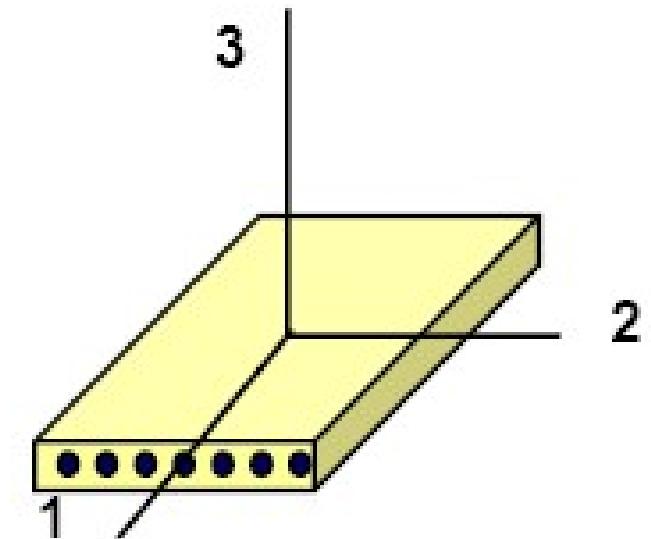
b)



# Equivalent Homogeneous Material

## Important Assumption:

- We assume that the fiber and matrix properties can be smeared and represented as an equivalent homogenous material with orthotropic material properties
- This allows us to develop the stress strain behavior of the material making the structural level response tractable
- Otherwise we would have to deal with micromechanics



# **Unidirectional Lamina (UD-Lamina)**

## **Stiffness of a UD-Lamina**

# Orthotropic Lamina – Symmetric Compliance Matrix

Due to the 3 reciprocal relations, only 9 independent elastic constants are needed

$$S_{11} = \frac{I}{E_{11}} \quad S_{12} = \frac{-\nu_{12}}{E_{11}} \quad S_{13} = \frac{-\nu_{13}}{E_{11}}$$

$$S_{22} = \frac{I}{E_{22}} \quad S_{23} = \frac{-\nu_{23}}{E_{22}} \quad S_{33} = \frac{I}{E_{33}}$$

$$S_{44} = \frac{I}{G_{23}} \quad S_{55} = \frac{I}{G_{13}} \quad S_{66} = \frac{I}{G_{12}}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix}$$

Zero entries in the upper right and lower left portions of the compliance matrix characterize orthotropic behavior

# Orthotropic Lamina – Symmetric Stiffness Matrix

The inverse relations are...

$$C_{11} = \frac{S_{22}S_{33} - S_{23}S_{32}}{S} \quad C_{12} = \frac{S_{13}S_{23} - S_{12}S_{33}}{S}$$

$$C_{22} = \frac{S_{33}S_{11} - S_{13}S_{31}}{S} \quad C_{13} = \frac{S_{12}S_{23} - S_{13}S_{22}}{S}$$

$$C_{33} = \frac{S_{11}S_{22} - S_{12}S_{21}}{S} \quad C_{23} = \frac{S_{12}S_{13} - S_{23}S_{11}}{S}$$

$$C_{44} = \frac{I}{S_{44}} \quad C_{55} = \frac{I}{S_{55}} \quad C_{66} = \frac{I}{S_{66}}$$

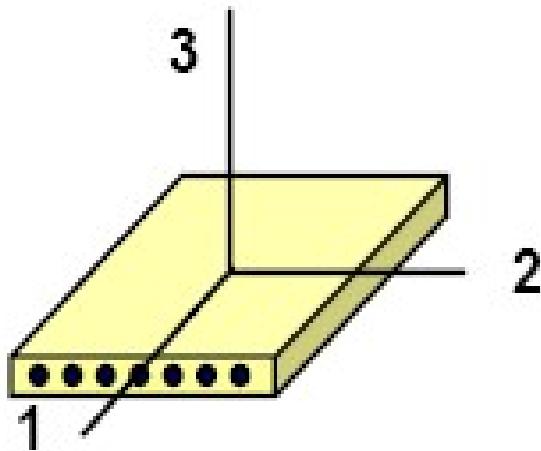
$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \tau_{23} \\ \tau_{13} \\ \tau_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{23} \\ \gamma_{13} \\ \gamma_{12} \end{bmatrix}$$

where

$$S = S_{11}S_{22}S_{33} - S_{11}S_{23}S_{32} - S_{22}S_{13}S_{31} - S_{33}S_{12}S_{21} + 2S_{12}S_{23}S_{31}$$

# Transversely Isotropic

Between orthotropic material behavior and isotropic material behavior is transversely isotropic behavior. We assume properties in the 2 and 3 directions are similar. We assume that...



$$E_{22} = E_{33} \quad \nu_{12} = \nu_{13} \quad G_{12} = G_{13}$$

$$\text{and } G_{23} = \frac{E_{22}}{2(1 + \nu_{23})}$$

# Transversely Isotropic

If we assume that properties in the 2 and 3 directions are similar, we arrive at the compliance matrix...

$$\begin{aligned} S_{11} &= \frac{l}{E_{11}} & S_{12} &= \frac{-\nu_{12}}{E_{11}} & S_{22} &= \frac{l}{E_{22}} & S_{23} &= \frac{-\nu_{23}}{E_{22}} \\ S_{33} &= \frac{l}{E_{33}} & S_{44} &= \frac{l}{G_{23}} = \frac{2(1+\nu_{23})}{E_{22}} & S_{55} &= \frac{l}{G_{12}} \end{aligned}$$

$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix}$$

We now have 5 independent material properties  
 $E_{11}, E_{22}, \nu_{12}, \nu_{23}, G_{12}$

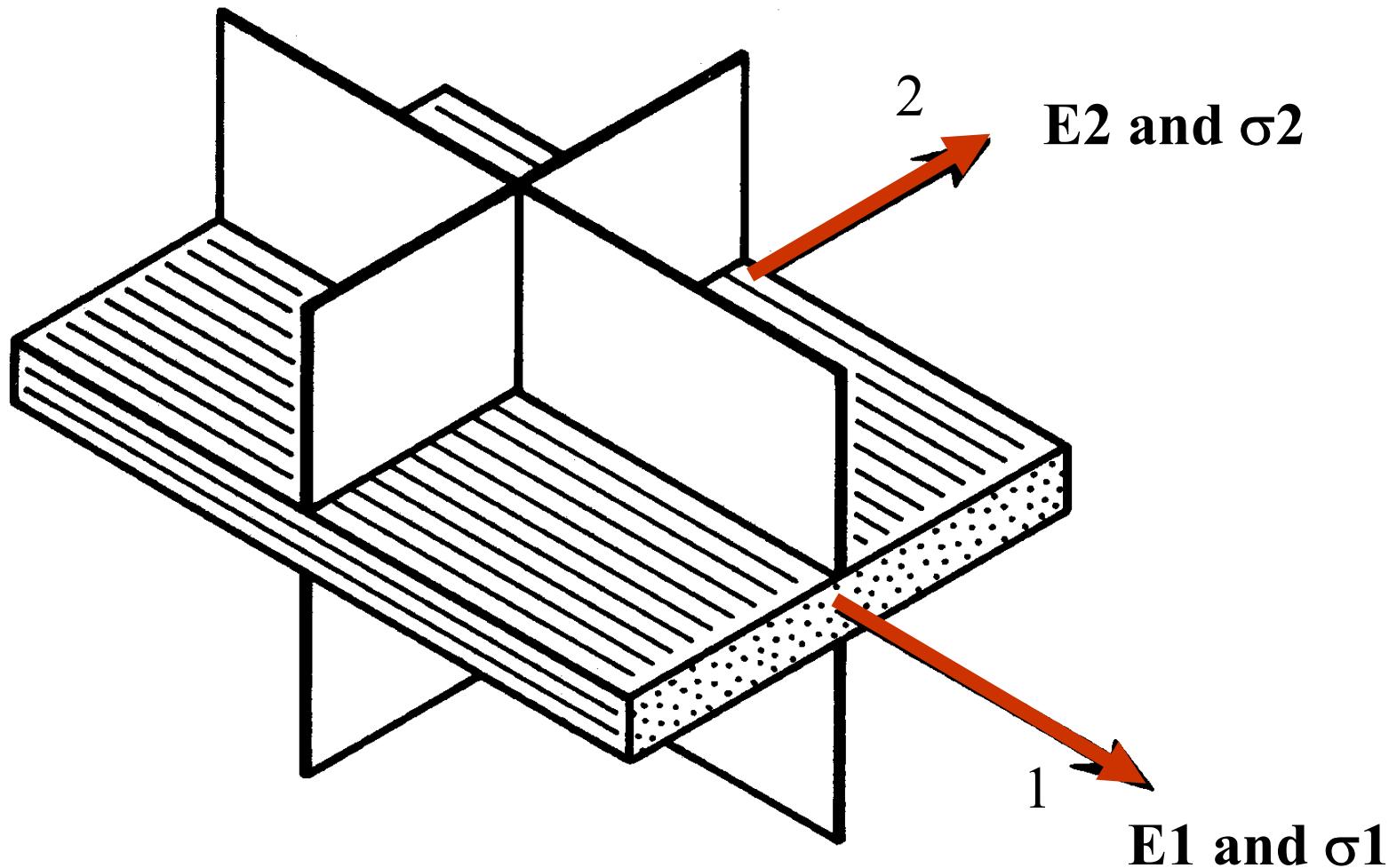
# Transversely Isotropic

If we assume that properties in the 2 and 3 directions are similar, we arrive at the stiffness matrix...

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{22} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix}$$

*Transversely Isotropic:* a material which contains a plane in which the mechanical properties are equal in all directions

## Symmetrical planes of a transverse isotropic UD-Lamina



# Plane Stress: Out-of-Plane Strains

$$\sigma_{33} = \tau_{13} = \tau_{23} = 0$$

As a result the out-of-plane strains are...

$$\gamma_{13} = \gamma_{23} = 0$$

$$\epsilon_{33} = S_{13}\sigma_{11} + S_{23}\sigma_{22}$$

Note that the out-of-plane normal strain  $\epsilon_{33}$  is not zero, as a result of the Poisson's ratios  $v_{13}$ ,  $v_{23}$ , acting through the  $S_{13}$  &  $S_{23}$ !

# Reduced Compliance Matrix

The Reduced Compliance Matrix is a result of our assumption of transverse isotropy and plane stress...

$$\begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \gamma_{12} \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \tau_{12} \end{Bmatrix}$$

Transversely Isotropic

$$S_{11} = \frac{I}{E_{11}} \quad S_{12} = \frac{-\nu_{12}}{E_{11}} = \frac{-\nu_{21}}{E_2}$$

$$S_{22} = \frac{I}{E_{22}} \quad S_{66} = \frac{I}{G_{12}}$$

4 independent material properties  
 $E_{11}, E_{22}, \nu_{12}, G_{12}$

Why ??

Isotropic

$$S_{11} = S_{22} = \frac{I}{E} \quad S_{12} = \frac{-\nu}{E}$$
$$S_{66} = \frac{I}{G} = \frac{2(I+\nu)}{E}$$

2 independent material properties  
 $E, \nu \text{ or } G, \nu \text{ or } E, G$

## Strain-Stress relation of a UD-Lamina, Plane stress

$$\varepsilon_1 = \frac{1}{E_1} \sigma_1 - \frac{\nu_{21}}{E_2} \sigma_2$$

$$\varepsilon_2 = -\frac{\nu_{12}}{E_1} \sigma_1 + \frac{1}{E_2} \sigma_2$$

$$\gamma_{12} = \frac{1}{G_{12}} \tau_{12}$$

## Stress-Strain relation of a UD-Lamina, Plane stress

## Inversion of the Compliance Matrix

$$\sigma_1 = \frac{E_1}{1 - \nu_{21}\nu_{12}} \varepsilon_1 + \frac{\nu_{12}E_2}{1 - \nu_{21}\nu_{12}} \varepsilon_2$$

$$\sigma_2 = \frac{\nu_{21}E_1}{1 - \nu_{21}\nu_{12}} \varepsilon_1 + \frac{E_2}{1 - \nu_{21}\nu_{12}} \varepsilon_2$$

$$\tau_{12} = G_{12} \gamma_{12}$$

# Stiffness Matrix Q of a UD-Lamina

	$\varepsilon_1$	$\varepsilon_2$	$\gamma_{12}$
$\sigma_1$	$Q_{11}$	$Q_{12}$	
$\sigma_2$	$Q_{21}$	$Q_{22}$	
$\tau_{12}$			$Q_{66}$

## Stiffnesses $Q_{ij}$ of a UD-Lamina

$$Q_{11} = \frac{E_1}{1 - \nu_{21}\nu_{12}}; Q_{22} = \frac{E_2}{1 - \nu_{21}\nu_{12}}; Q_{66} = G_{12}$$

$$Q_{21} = \frac{\nu_{21}E_1}{1 - \nu_{21}\nu_{12}}; Q_{12} = \frac{\nu_{12}E_2}{1 - \nu_{21}\nu_{12}}$$

# Symmetry ?

Stiffness matrix Q:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \quad \left( \frac{N}{mm^2} \right)$$

Compliance matrix S:

**Symmetry ?**

$$S = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \quad \left( \frac{mm^2}{N} \right)$$

## Symmetry ??

$$Q_{12} = Q_{21}$$

and

$$S_{12} = S_{21}$$

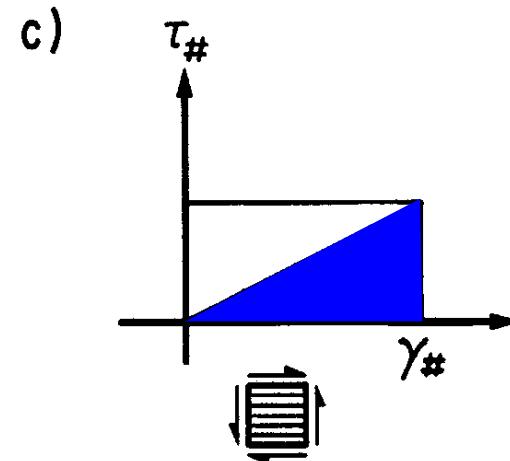
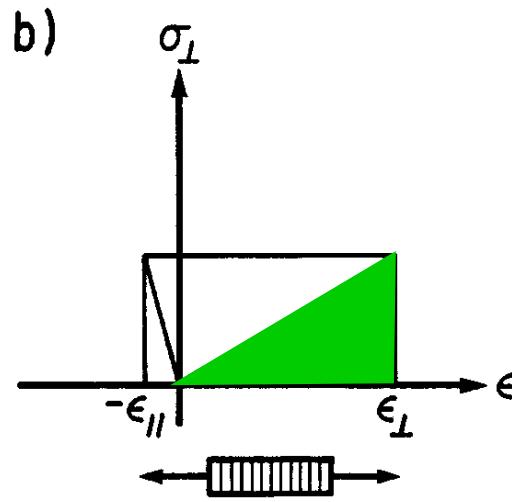
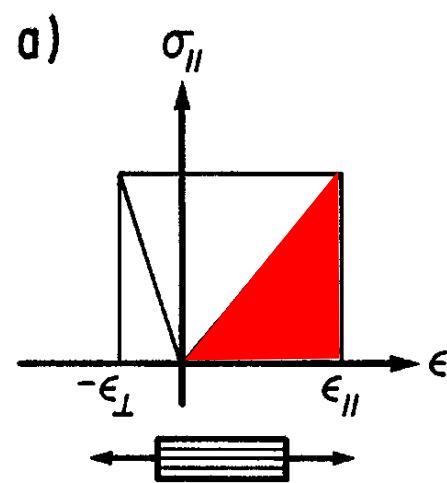
???

## Coupling Terms

# Elastic Energy

The stored elastic energy in the UD-Lamina is:

$$W = \frac{1}{2} [\underline{\sigma_1 \epsilon_1} + \underline{\sigma_2 \epsilon_2} + \underline{\tau_{12} \gamma_{12}}]$$



Replace the strains with stresses using the compliance matrix as follow:

	$\sigma_1$	$\sigma_2$	$\tau_{12}$
$\epsilon_1$	$S_{11}$	$S_{12}$	
$\epsilon_2$	$S_{21}$	$S_{22}$	
$\gamma_{12}$			$S_{66}$

We obtain :

$$W = \frac{1}{2} [S_{11}\sigma_1^2 + (S_{12} + S_{21})\sigma_1\sigma_2 + S_{22}\sigma_2^2 + S_{12}\tau_{12}^2]$$

Partial differentiation provides the following strain-stress relation:

$$\frac{\partial W}{\partial \sigma_1} = \left[ S_{11}\sigma_1 + \frac{1}{2}(S_{12} + S_{21})\sigma_2 \right] = \epsilon_1$$

$$\frac{\partial W}{\partial \sigma_2} = \left[ \frac{1}{2}(S_{12} + S_{21})\sigma_1 + S_{22}\sigma_2 \right] = \epsilon_2$$

Compare the equations with the strain-stress relation through compliance matrix

	$\sigma_1$	$\sigma_2$	$\tau_{12}$
$\varepsilon_1$	$S_{11}$	$S_{12}$	
$\varepsilon_2$	$S_{21}$	$S_{22}$	
$\gamma_{12}$			$S_{66}$

we obtain:

$$S_{12} = S_{21}$$

If we use the stiffness matrix

	$\varepsilon_1$	$\varepsilon_2$	$\gamma_{12}$
$\sigma_1$	$Q_{11}$	$Q_{12}$	
$\sigma_2$	$Q_{21}$	$Q_{22}$	
$\tau_{12}$			$Q_{66}$

and replace stresses with strains, we obtain in a similar way the following equation:

$$Q_{12} = Q_{21}$$

or following equation is obtained for engineering constants:

$$\nu_{21}E_1 = \nu_{12}E_2$$

Or:

$$\frac{\nu_{21}}{\nu_{12}} = \frac{E_2}{E_1}$$

# Definitions:

- **Orthotropy**: Anisotropy with 3 orthogonal planes of symmetry.

9 independent constants

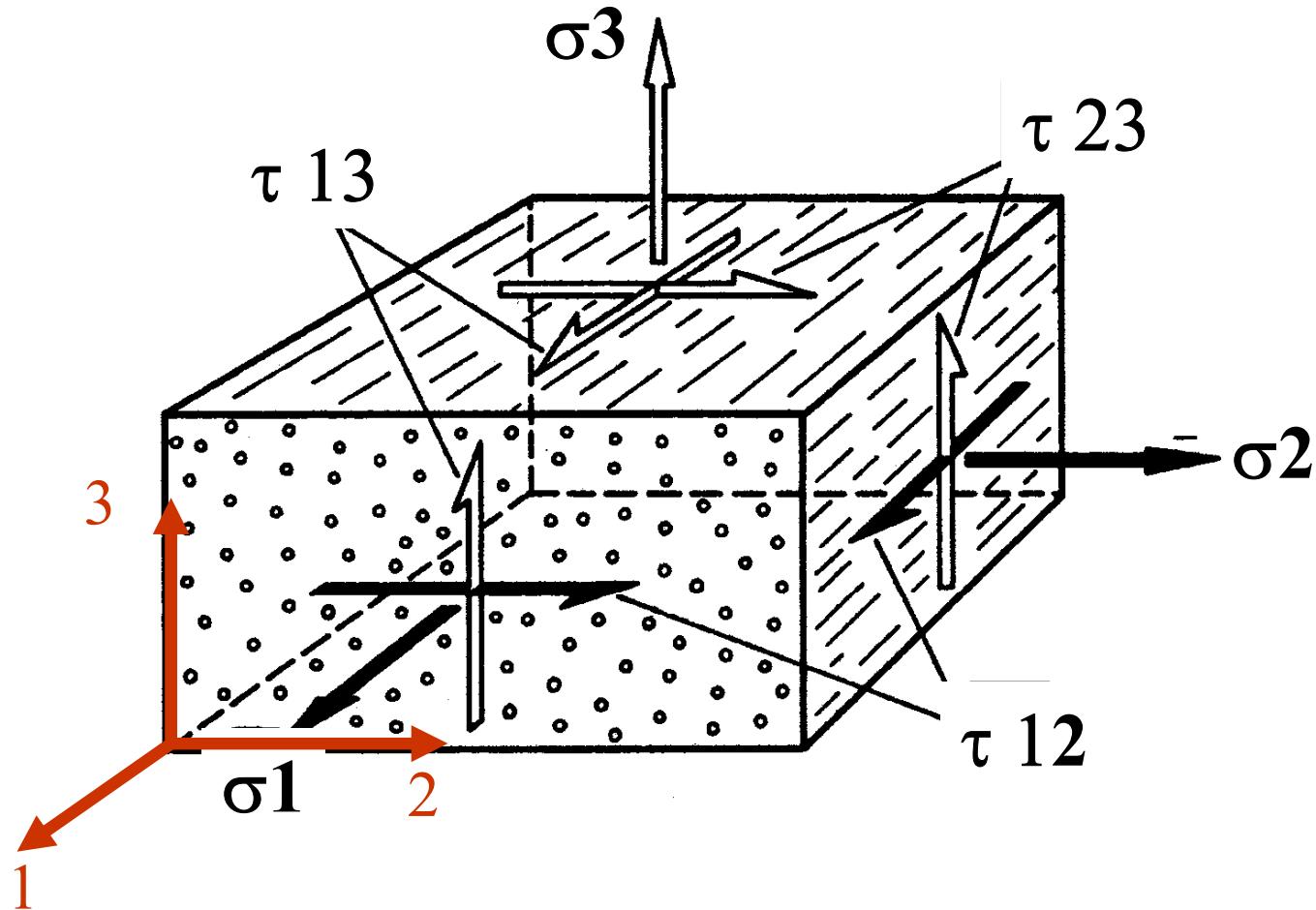
- **Transverse Isotropy**: Orthotropy with a plane at which there is Isotropy

5 independent constants

- **Transverse Isotropy and Plane Stress**:

4 independent constants

## Stress State of a Unidirectional (UD) Lamina: Transverse Isotropy, Homogeneous



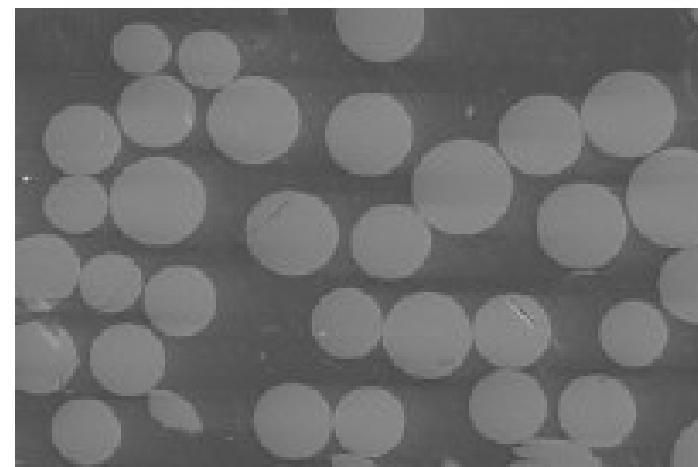
**Plane Stress:**  $\sigma_1 \quad \sigma_2 \quad \tau_{12}$

## Independent Elasticity Constants

If there are more symmetry conditions, there will be further reductions in the number of constants. For a **cross-ply** laminate with  $E_1=E_2$ , there are **3** and for an **isotropic** material (for example mat-laminate with randomly distributed fibers) **2** independent constants.

# Estimating Elastic Composite Properties

- Assumes...
  - The fiber and matrix are bonded
  - Pure modes (extension/shear)
  - Fiber packing – representative volume
- Approaches
  - FEA
  - Mechanics of materials
  - Semi empirical equations



Elasticity constants of a UD-Lamina are dependent on the following

$E_F$  = **Fiber E-Modulus**

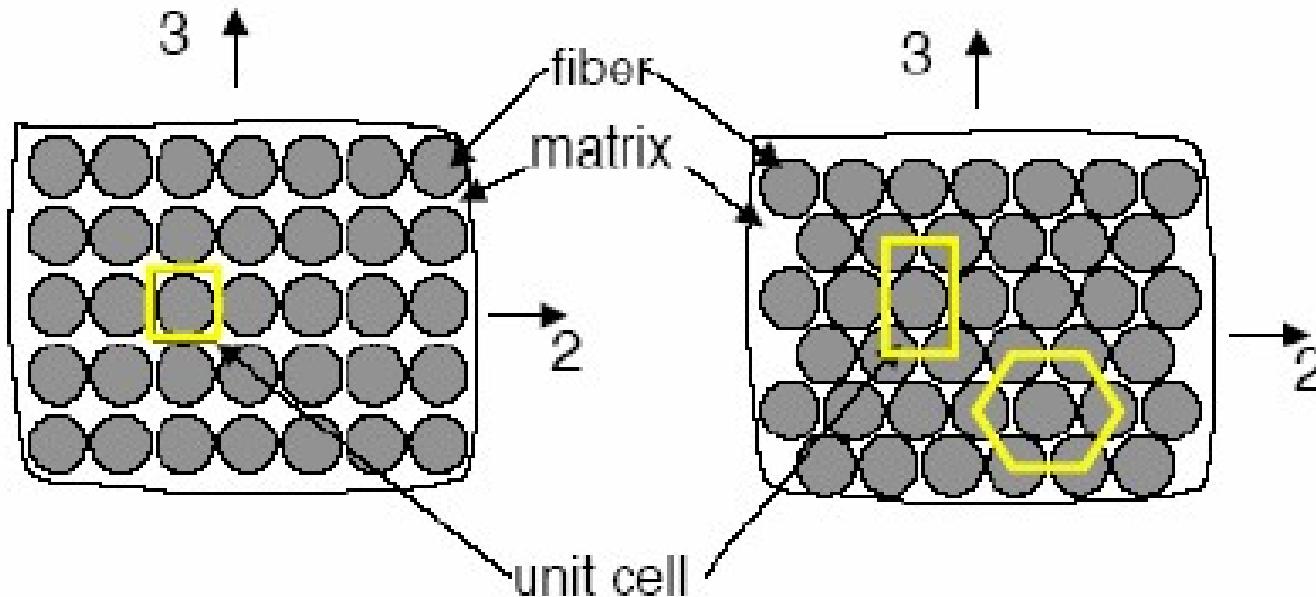
$\nu_F$  = **Fiber Poisson's Ratio**

$E_M$  = **Matrix E-Modulus**

$\nu_M$  = **Matrix Poisson's Ratio**

$\phi$  = **Fiber Volume Fraction**

# FEA Examination



- Idealized packing
- Representative volume
- Boundary conditions applied
- Stresses/Strains integrated to derive properties
- Local stress states/damage

# **Mechanics of Materials Approaches to Elastic Property Estimates**

# Rule of Mixtures

**Equal strain**



$$\varepsilon = \varepsilon_f = \varepsilon_m$$

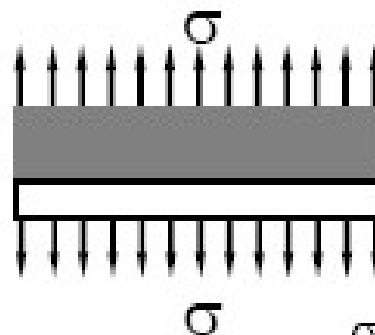
$$\begin{aligned}\sigma &= \sigma_f V_f + \sigma_m (1 - V_f) \\ &= E_f \varepsilon V_f + E_m \varepsilon (1 - V_f)\end{aligned}$$

$$E_{11} = E_f^f V^f + E_m (1 - V^f)$$

Similar for  $\nu_{12}$

$$\nu_{12} = \nu_{12}^f V^f + \nu_m (1 - V^f)$$

**Equal stress**



$$\sigma = \sigma_f = \sigma_m$$

$$\varepsilon = \varepsilon_f V_f + \varepsilon_m (1 - V_f)$$

$$\varepsilon = \frac{\sigma_f}{E_f} V_f + \frac{\sigma_m}{E_m} (1 - V_f)$$

$$\frac{1}{E_{22}} = \frac{V^f}{E_f^f} + \frac{(1 - V^f)}{E^m}$$

$$\frac{1}{G_{12}} = \frac{V^f}{G_{12}^f} + \frac{(1 - V^f)}{G^m}$$

Not so good for  $E_2$  &  $G_{12}$

The longitudinal E-Modulus (parallel to the fiber direction) can be derived from the following so called rule of mixture:

$$E_1 = \Phi_F E_F + (1 - \Phi_F) E_M$$

where  $\phi_F$  = fiber volume content of the UD-Lamina

The major poisson's ratio  $\nu_{12}$  caused by longitudinal stresses following the rule of mixture is:

$$\nu_{12} = \Phi_F \nu_F + (1 - \Phi_F) \nu_M$$

And the minor poisson's ratio is:

$$\nu_{21} = \nu_{12} \frac{E_2}{E_1}$$

# **Semi Empirical Equations Based on Experiments**

The transverse E-modulus for **isotropic** fibers according to ,Puck' can be obtained:

$$E_2 = E_M^o \frac{1 + 0.85\Phi_F^2}{\Phi_F E_M^o / E_F + (1 - \Phi_F)^{1.25}}$$

$$E_M^o = \frac{E_M}{1 - \nu_M^2}$$

According to ,Puck' for **isotropic** fibers:

$$G_{12} = G_M \frac{\left(1 + 0.60\Phi_F^{0.5}\right)}{\Phi_F G_M / G_F + \left(1 - \Phi_F\right)^{1.25}}$$

According to ,Fürster' and ,Schneider' for **isotropic** fibers:

$$E_2 = E_M^o \frac{1}{\Phi_F E_M^o / E_F + (1 - \Phi_F)^{1.45}}$$

$$E_M^o = \frac{E_M}{1 - \nu_M^2}$$

According to ,Förster' and ,Schneider' for **isotropic** fibers:

$$G_{12} = G_M \frac{\left(1 + 0.4\Phi_F^{0.5}\right)}{\Phi_F G_M / G_F + \left(1 - \Phi_F\right)^{1.45}}$$

Following ,Tsai' for **isotropic** fibers:

$$E_2 = \frac{E_M (1 + \xi - \eta \Phi_F)}{1 - \eta \Phi_F} \quad \text{where}$$

$$\eta = \frac{E_F / E_M - 1}{E_F / E_M + \xi} \quad \text{and} \quad \xi = 2$$

Following ,Tsai' for **isotropic** fibers:

$$G_{12} = \frac{G_M (1 + \xi \quad \eta \quad \Phi_F)}{1 - \eta \quad \Phi_F} \quad \text{where}$$

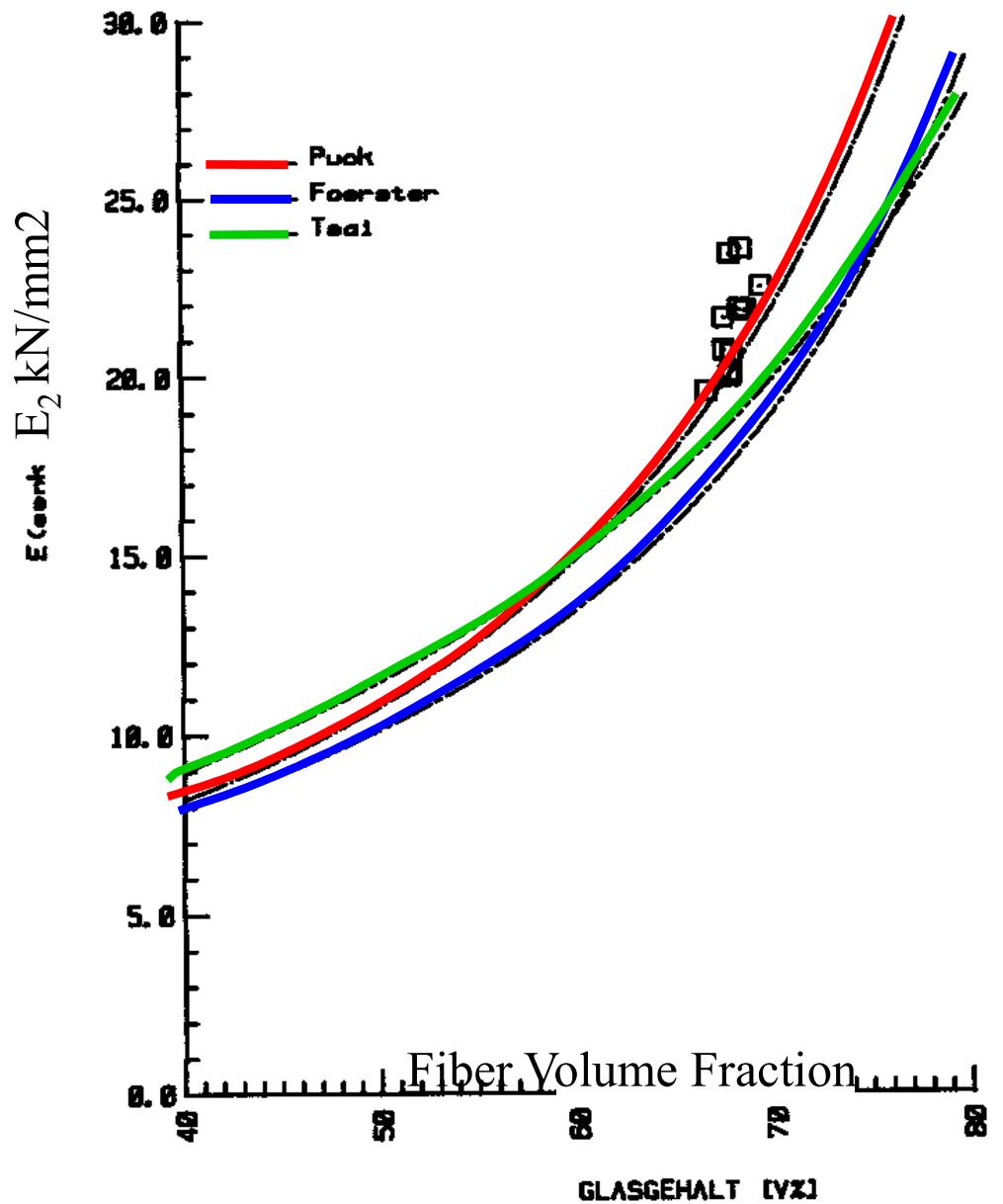
$$\eta = \frac{G_F / G_M - 1}{G_F / G_M + \xi} \quad \text{and} \quad \xi = 1$$

## Comparisons for $E_2$

$$E_2 = E_M^o \frac{1 + 0.85\Phi_F^2}{\Phi_F E_M^o / E_F + (1 - \Phi_F)^{1.25}}$$

$$E_2 = E_M^o \frac{1}{\Phi_F E_M^o / E_F + (1 - \Phi_F)^{1.45}}$$

$$E_2 = \frac{E_M(1 + \xi - \eta \Phi_F)}{1 - \eta \Phi_F}$$

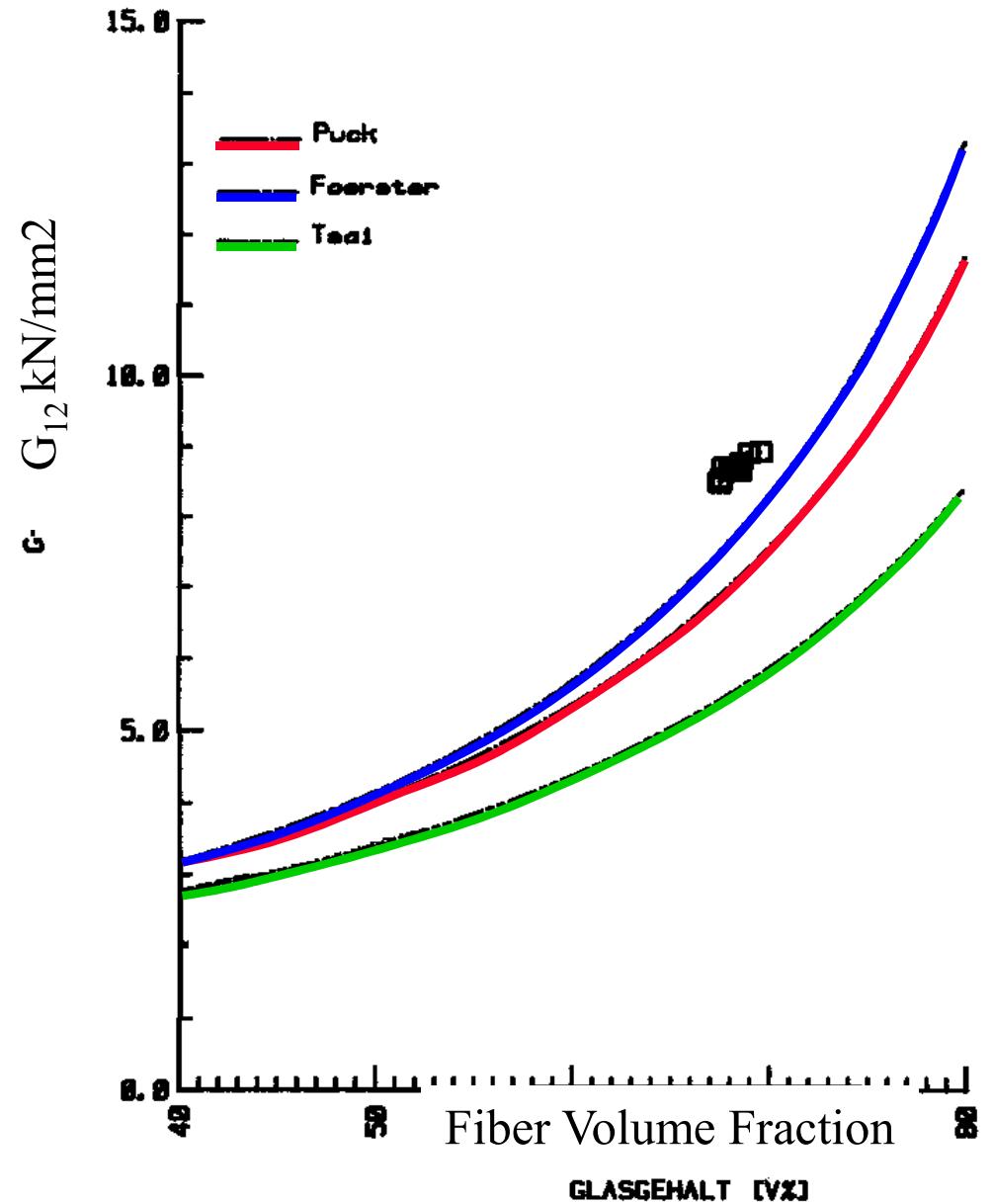


## Comparisons for $G_{12}$

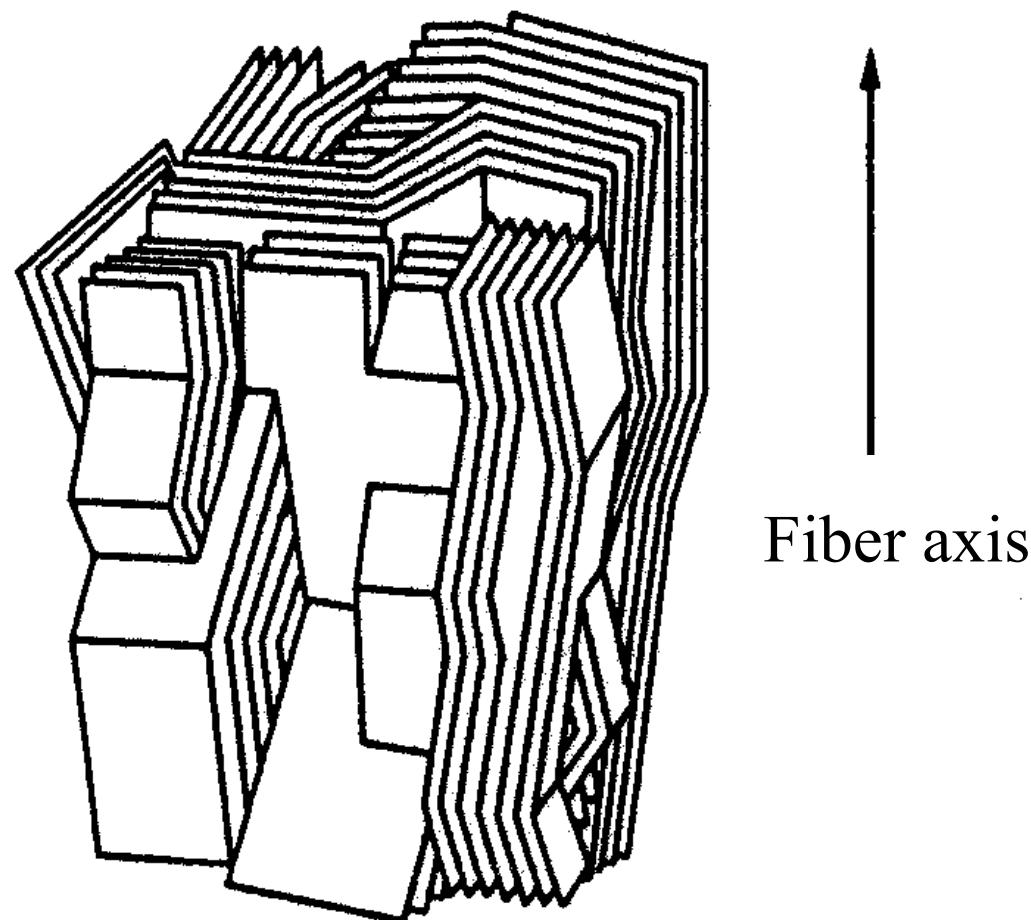
$$G_{12} = G_M \frac{(1 + 0.60\Phi_F^{0.5})}{\Phi_F G_M / G_F + (1 - \Phi_F)^{1.25}}$$

$$G_{12} = G_M \frac{(1 + 0.4\Phi_F^{0.5})}{\Phi_F G_M / G_F + (1 - \Phi_F)^{1.45}}$$

$$G_{12} = \frac{G_M(1 + \xi)}{1 - \eta} \frac{\eta}{\Phi_F}$$



## Anisotropic fibers



For anisotropic fibers like C-fibers following equations can be applied:

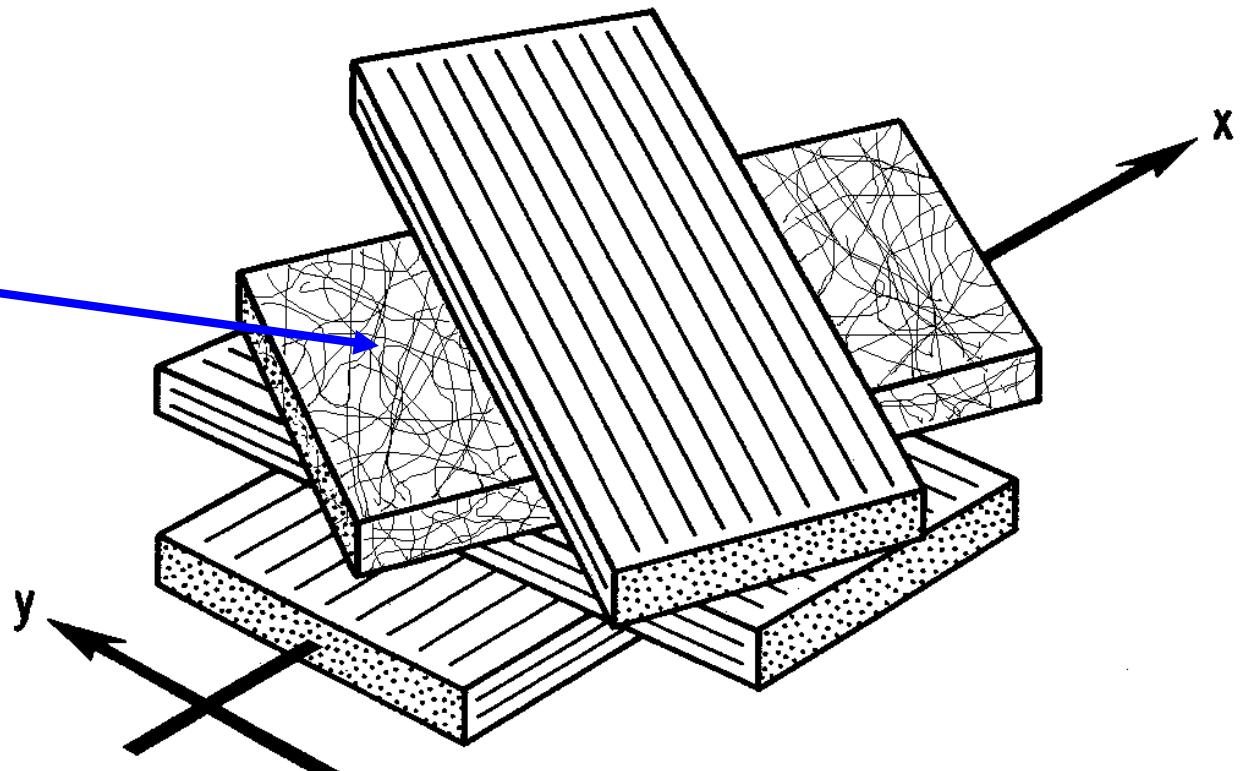
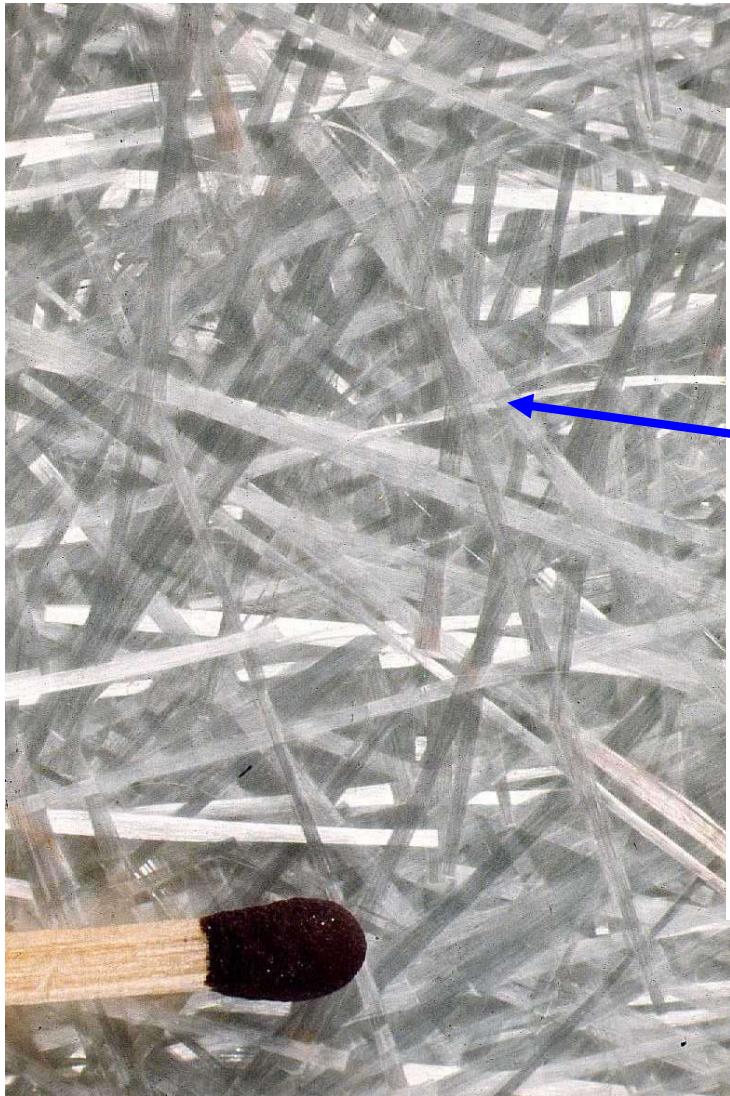
$$E_2 = \frac{E_M^o \left(1 + \Phi_F^3\right)}{\left(1 - \Phi_F\right)^{0.75} + 6 \Phi_F E_M^o / E_F}$$

$$E_M^o = \frac{E_M}{1 - \nu_M^2}$$

and

$$G_{12} = \frac{G_M \left( 1 + 0.25 \Phi_F^{0.5} \right)}{\left( 1 - \Phi_F \right)^{1.25} + 1.25 \Phi_F G_M / G_F}$$

## Glass-Mat Lamina



Following equations can be applied to E-Glass Mat Lamina according to Puck :

$$E \approx 29'630 - 4'710 \Phi_F^2 + 3'920$$

$$\nu \approx 0.34 - 0.075 \Phi_F$$

$$G \approx 10'970 + 1'370 \Phi_F$$

# Elasticity constants of some UD-Laminas

Lamina type	T 300/5208	B (4)/5505	AS/3501	Scotchply 1002	Kevlar 49 / Epoxy
Fiber	C-Fibers from Toray	Boron	C –Fibers from Hercules	E-Glass	Aramid from E.I. Dupont de Nemours
Matrix	EP from Narmco	EP-Prepreg from Avco	EP-Prepreg from Hercules	EP-Prepreg from 3M	EP
Fiber volume fraction $\Phi$ (%)	70	50	66	45	60
Density (g/cm <sup>3</sup> )	1.6	2.0	1.6	1.8	1.46
$E_1$ (N/mm <sup>2</sup> )	181'000	204'000	138'000	38'600	76'000
$E_2$ (N/mm <sup>2</sup> )	10'300	18'500	8'960	8'270	5'500
$v_{12}$	0.28	0.23	0.30	0.26	0.34
$G_{12}$ (N/mm <sup>2</sup> )	7'170	5'590	7'100	4'140	2'300

## ...Elasticity constants of some UD-Laminas

Lamina type	T 300/5208	B (4)/5505	AS/3501	Scotchply 1002	Kevlar 49 / Epoxy
$E_1$ (N/mm <sup>2</sup> )	181'000	204'000	138'000	38'600	76'000
$E_2$ (N/mm <sup>2</sup> )	10'300	18'500	8'960	8'270	5'500
$\nu_{12}$	0.28	0.23	0.30	0.26	0.34
$G_{12}$ (N/mm <sup>2</sup> )	7'170	5'590	7'100	4'140	2'300
$S_{11}$ (mm <sup>2</sup> /N)	$5.525 \cdot 10^{-6}$	$4.902 \cdot 10^{-6}$	$7.246 \cdot 10^{-6}$	$25.91 \cdot 10^{-6}$	$13.16 \cdot 10^{-6}$
$S_{22}$ (mm <sup>2</sup> /N)	$97.09 \cdot 10^{-6}$	$54.05 \cdot 10^{-6}$	$111.6 \cdot 10^{-6}$	$120.9 \cdot 10^{-6}$	$181.8 \cdot 10^{-6}$
$S_{12}$ (mm <sup>2</sup> /N)	$-1.547 \cdot 10^{-6}$	$-1.128 \cdot 10^{-6}$	$-2.174 \cdot 10^{-6}$	$-6.744 \cdot 10^{-6}$	$-4.474 \cdot 10^{-6}$
$S_{33}$ (mm <sup>2</sup> /N)	$139.5 \cdot 10^{-6}$	$172.7 \cdot 10^{-6}$	$140.8 \cdot 10^{-6}$	$241.5 \cdot 10^{-6}$	$434.8 \cdot 10^{-6}$
$Q_{11}$ (N/mm <sup>2</sup> )	181'800	205'000	138'000	39'160	76'640
$Q_{22}$ (N/mm <sup>2</sup> )	10'340	18'580	9'013	8'392	5'546
$Q_{12}$ (N/mm <sup>2</sup> )	2'897	4'275	2'704	2'182	1'886
$Q_{33}$ (N/mm <sup>2</sup> )	7'170	5'790	7'100	4'140	2'300

## Thermal properties of a UD-Lamina: Expansion coefficients (Book Geoff Eckold p59)

$$\alpha_1 = \alpha_F + \frac{\alpha_M - \alpha_F}{\Phi E_F + (1 - \Phi)E_M}$$

And (Book Geoff Eckold p59)

$$\alpha_2 = \alpha_F \Phi + \alpha_M (1 - \Phi) + \nu_F \alpha_F \Phi + \\ \nu_M \alpha_M (1 - \Phi) - [\nu_F \Phi + \nu_M (1 - \Phi)] \alpha_1$$

where:

$\alpha_{F_1}$  = Thermal expansion coefficient of fibers in fiber longitudinal direction

$\alpha_{F_2}$  = Thermal expansion coefficient of fibers perpendicular to fiber longitudinal direction

$\alpha_M$  = Thermal expansion coefficient of matrix

$\nu_M$  = Poisson's ratio of matrix

$E_{F_1}$  = Elasticity modulus of fibers in fiber longitudinal direction

$E_{F_2}$  = Elasticity modulus of fibers perpendicular to fiber longitudinal direction

$E_M$  = Elasticity modulus of matrix

$\Phi_F$  = Fiber volume content

# Failure Theories for a UD-Lamina

## Failure Theories for a UD-Lamina

Following simple criteria can be applied to examine the **fiber failure**:

$$\left( \frac{\sigma_1}{\sigma_{1\max}} \right) = 1$$

$\sigma_{1\max}$  = Failure stress of a UD-Lamina in fiber direction

$$\sigma_{1\max} = \sigma_{F\max} \Phi + (1 - \Phi) \sigma_{M\max}$$

## Matrix failure:

$$\left( \frac{\sigma_1}{\sigma_{1M\max}} \right)^2 + \frac{\sigma_2^2}{\sigma_{2C\max} \sigma_{2T\max}} + \frac{\sigma_{2C\max} - \sigma_{2T\max}}{\sigma_{2C\max} \sigma_{2T\max}} \sigma_2 + \left( \frac{\tau_{12}}{\tau_{12\max}} \right)^2 = 1$$

where

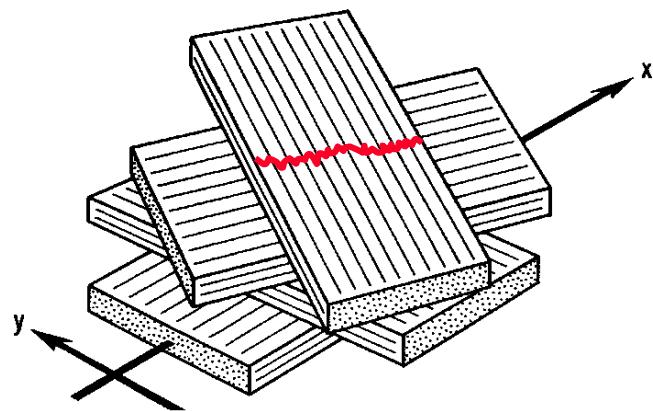
$$\sigma_{1M\max} = E_{1F} \epsilon_{M\max}$$

$\sigma_{2C\max}$  = Compression strength perpendicular to fiber direction

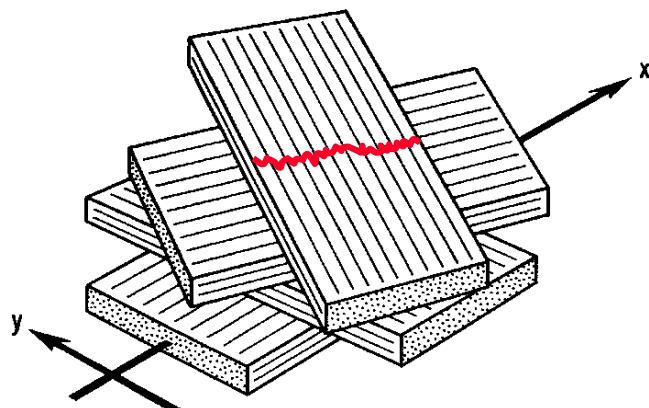
$\sigma_{2T\max}$  = Tensile strength perpendicular to fiber direction

$\tau_{12\max}$  = Shear strength

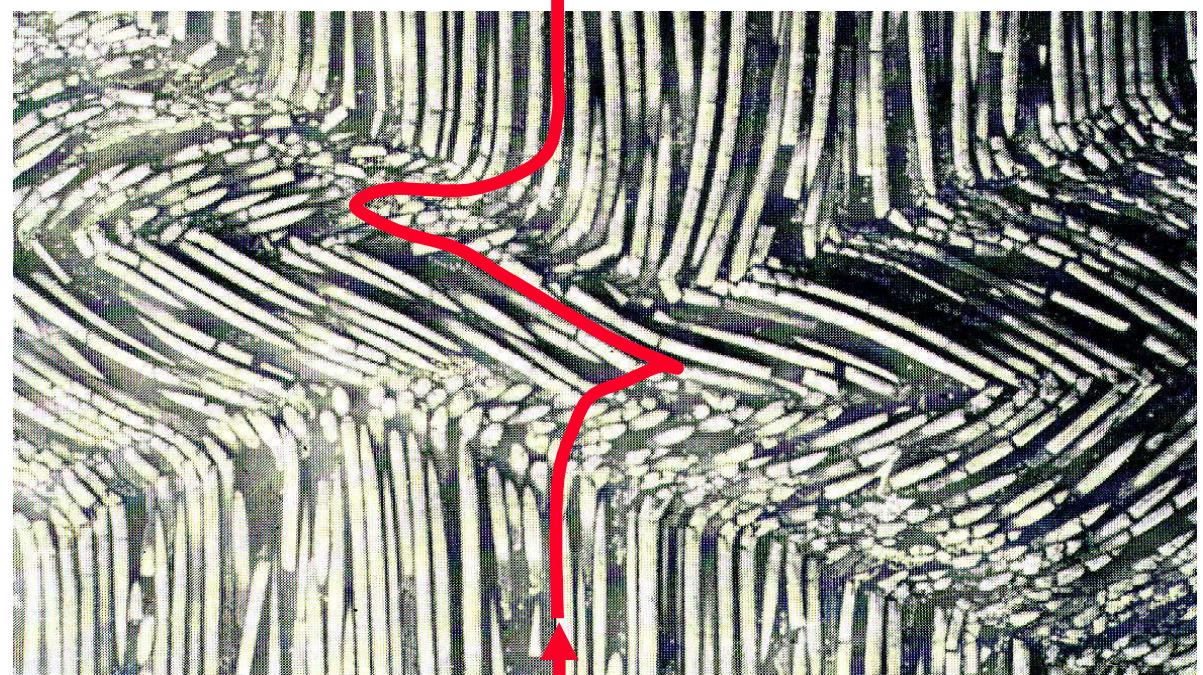
## Fiber failure due to tensile stress



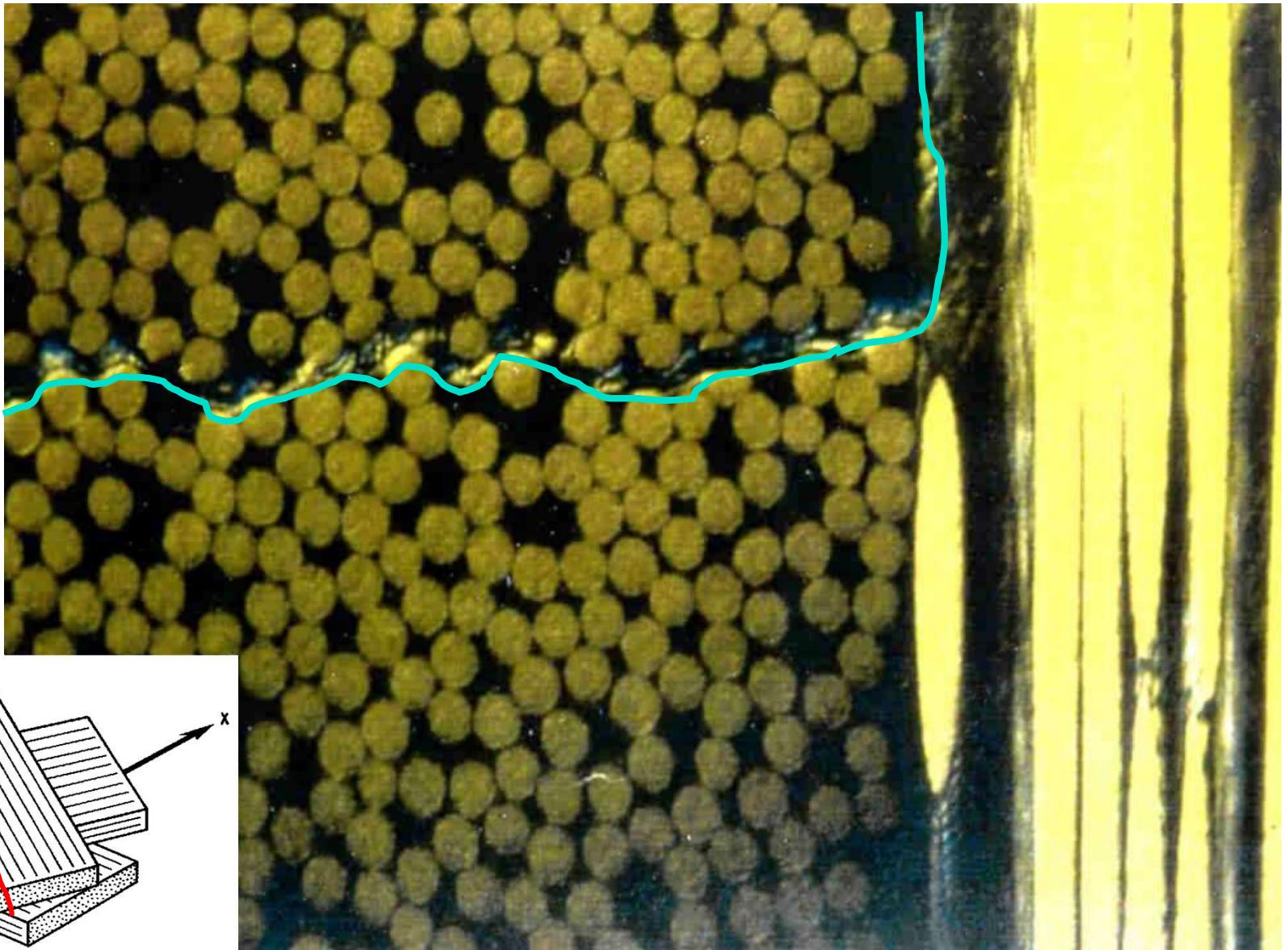
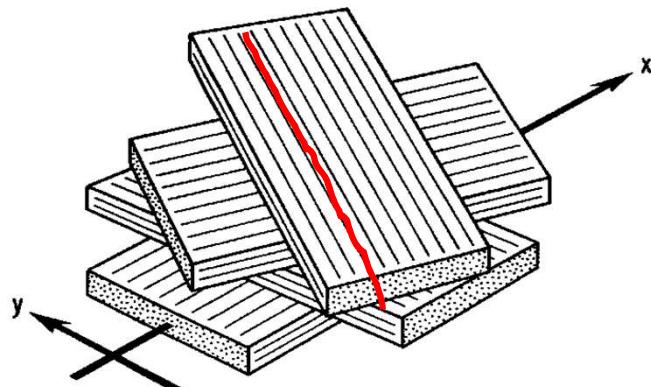
# Fiber failure due to compression- stresses



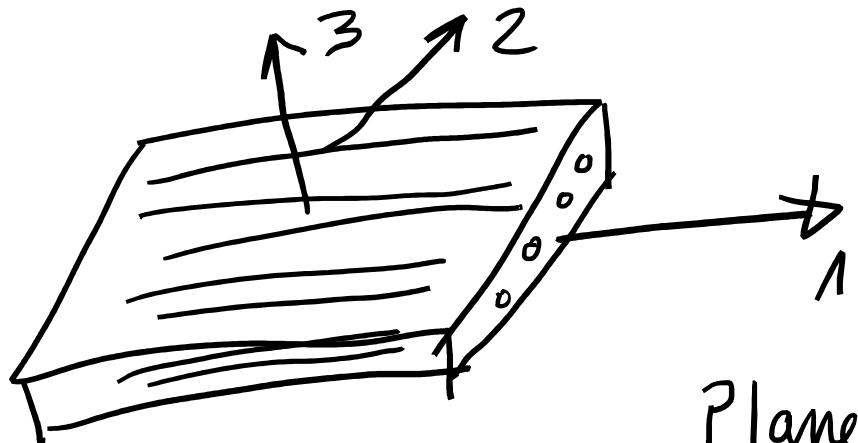
# Fiber failure due to compression- stresses



## Matrix failure



UD-Lamina



orthotropic

9 independent  
Const

$$E_{11}, E_{22}, E_{33}, G_{12}, G_{13}, G_{23}, \nu_{12}, \nu_{13}, \nu_{23}$$

Plane Stress  
4 Ind. Const

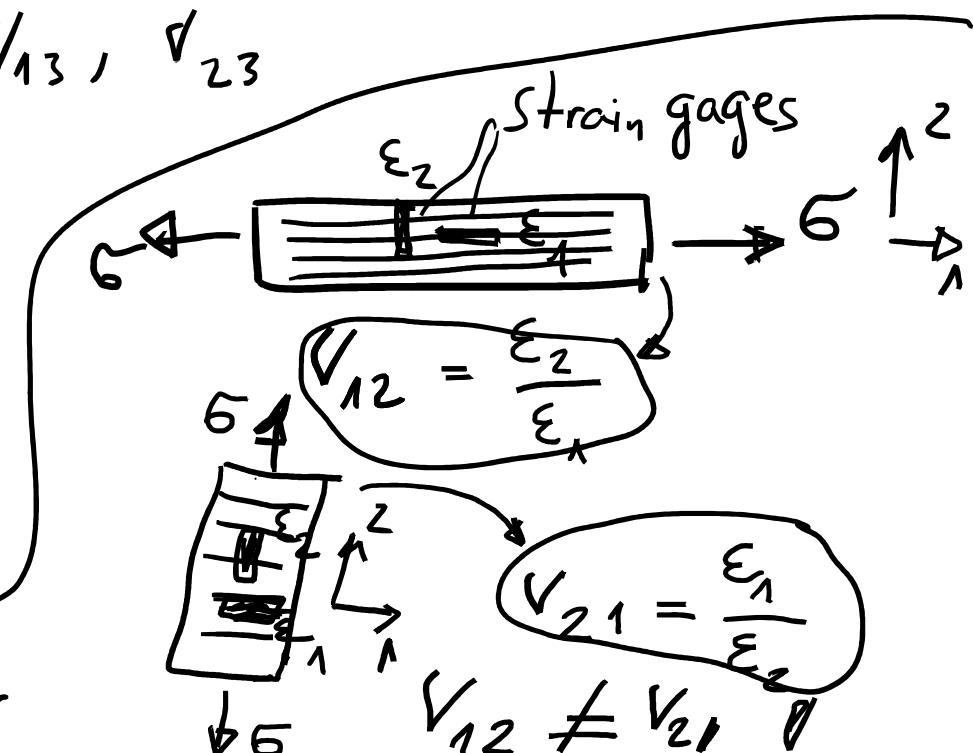
$$E_{11}, E_{22}, G_{12}, \nu_{12}$$

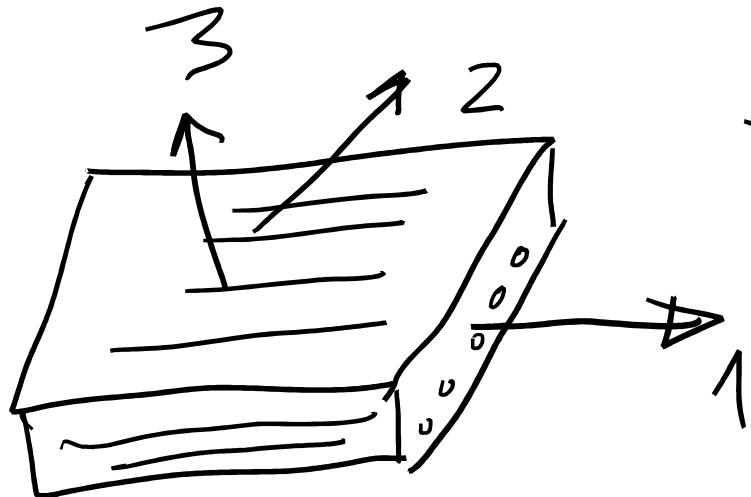
$\nu_{12}$ : Major Poisson ratio /  $\nu_{21}$ : Minor

transversal isotropic

5 Ind. Const.

$$E_{11}, E_{22} = E_{33}, G_{12}, G_{13}, G_{23}$$





UD - Lamina stiffness matrix

$$\begin{pmatrix} G_{11} \\ G_{22} \\ \tau_{12} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

$$Q_{12} = Q_{21} !$$

$$\left. \begin{array}{l} E_F, V_F \\ E_M, V_M \\ Q_F \end{array} \right\} \rightarrow \left. \begin{array}{l} E_{11}, E_{22} \\ G_{12}, \gamma_{12} \end{array} \right\} \rightarrow \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix}$$

Theoretical:

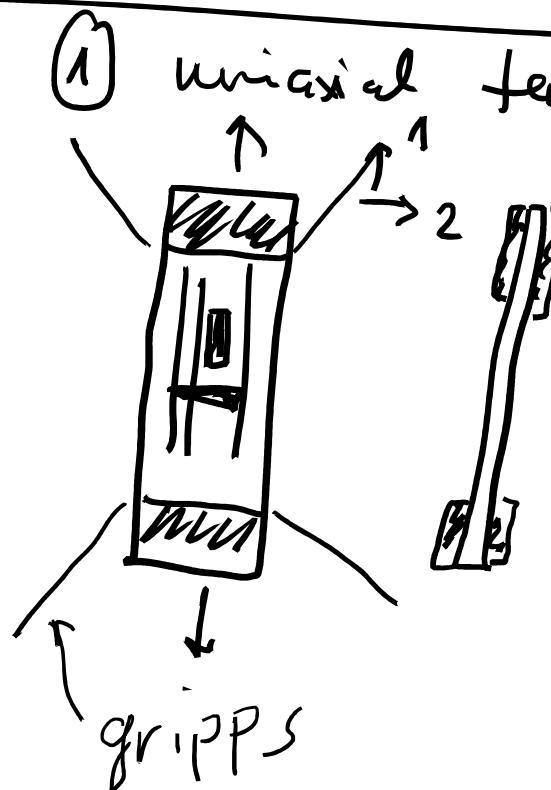
$$E_{11} = E_F \varphi_F + E_M (1 - \varphi_F)$$

$$\gamma_{12} = V_F \varphi_F + V_m (1 - \varphi_F)$$

$$E_{22} \quad \xrightarrow{\text{See semi empirical equations}}$$

# Experimental determination of UD-Lamina properties

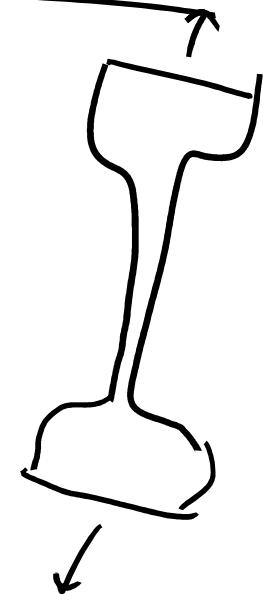
$E_{11}, E_{22}, \nu_{12}, G_{12}$



tensile test:  $E_{11}, \nu_{12}, \sigma_{11}$  failure

$$\begin{pmatrix} \sigma_{11} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 0 \end{pmatrix} \quad \text{or}$$

Measurement uncertainties:



Ahmadreza Hejazi

Subject: \_\_\_\_\_ Date: \_\_\_\_\_ Day: \_\_\_\_\_ Time: \_\_\_\_\_

①  $\begin{pmatrix} \sigma_{11} \\ \sigma_{22} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix} \cdot \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \end{pmatrix}$

②  $\nu_{12}, \epsilon_{11}, \epsilon_{22}$  Known

$$\nu_{12} = -\frac{\epsilon_{22}}{\epsilon_{11}} \quad \nu_{21} = \nu_{12} \frac{E_2}{E_1}$$

③  $\sigma_{11} = \frac{E_1}{1 - \nu_{21}\nu_{12}} \epsilon_{11} + \frac{\nu_{12} E_2}{1 - \nu_{21}\nu_{12}} \epsilon_{22}$

$$\sigma_1 = \frac{E_1}{1 - \nu_{12}^2 \frac{E_2}{E_1}} \epsilon_{11} + \frac{\nu_{12} E_2}{1 - \nu_{12}^2 \frac{E_2}{E_1}} (-\nu_{12} \epsilon_{11})$$

$$\sigma_1 = E_1 \epsilon_{11} \left( \frac{E_1 - \nu_{12}^2 E_2}{E_1 - \nu_{12}^2 E_2} \right) \rightarrow \boxed{\sigma_1 = E_1 \epsilon_{11}}$$

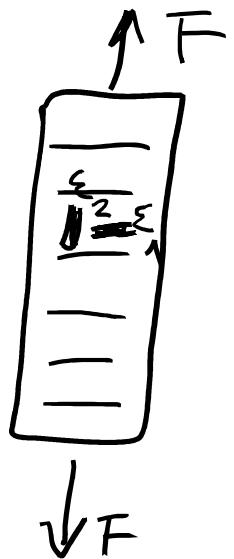
④

$$\frac{\nu_{21} E_1}{1 - \nu_{21} \nu_{12}} \epsilon_{11} + \frac{E_2}{1 - \nu_{21} \nu_{12}} \epsilon_{22} = 0$$

$$(\nu_{12} E_2) \epsilon_{11} + E_2 (-\nu_{12} \epsilon_{11}) = 0$$

$$0 = 0 \quad !!$$

② uniaxial tensile test :  $E_{22}$ ,  $\nu_{21}$ ,  $\epsilon_{22}$  failure



better

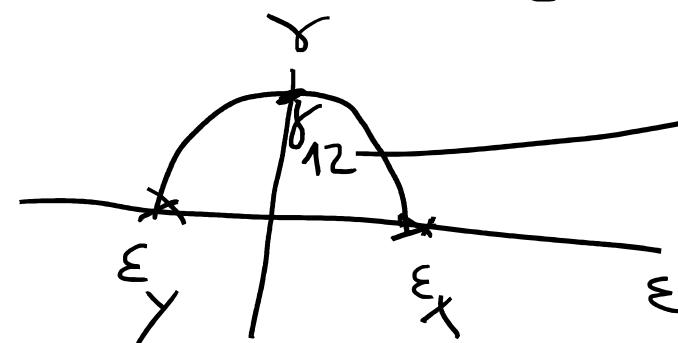


③

Torsion test :  $G_{12}$ ,  $\bar{\tau}_{12}$  failure



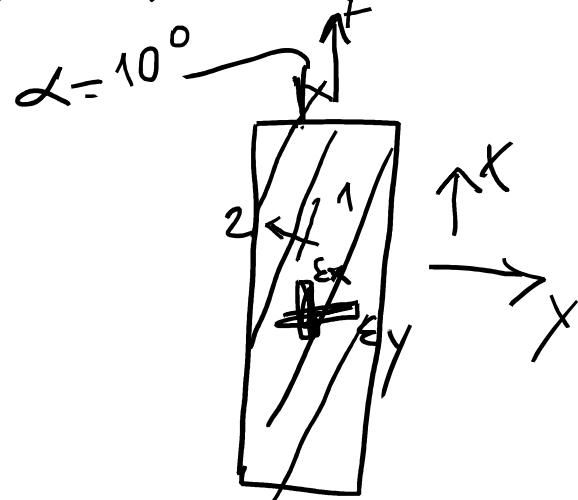
$\gamma_1$   
 $\gamma_2$



$$\bar{\tau}_{12} = G_{12} \cdot \gamma_{12}$$

$$\bar{\tau}_{12} = f(T, R, t)$$

④ off-axis tensile test :  $G_{12}$



$$\begin{pmatrix} \bar{\epsilon}_x \\ 0 \\ 0 \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} \bar{\epsilon}_{11} \\ \bar{\epsilon}_{22} \\ \bar{\gamma}_{12} \end{pmatrix}$$

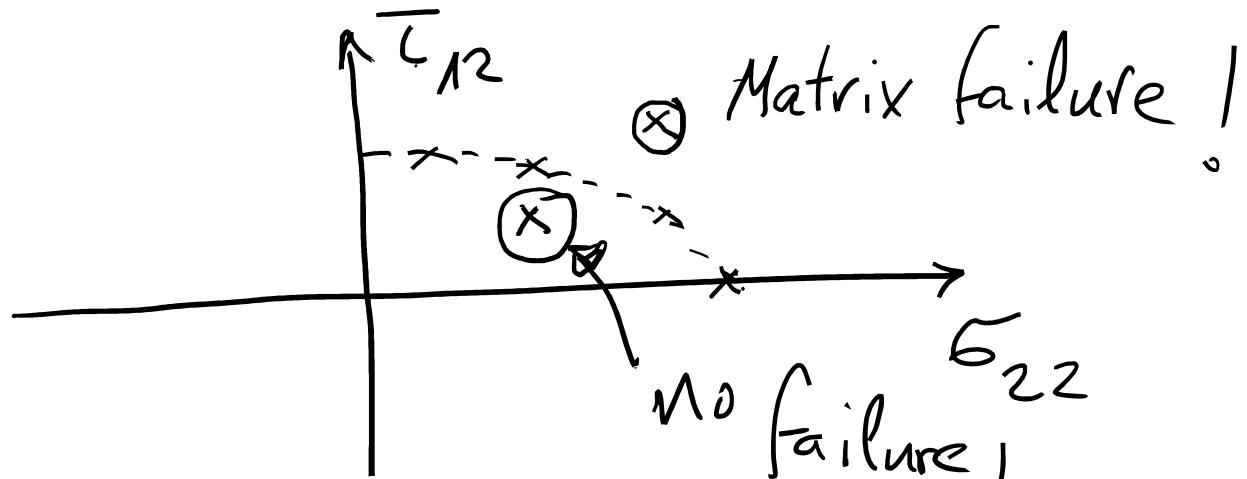
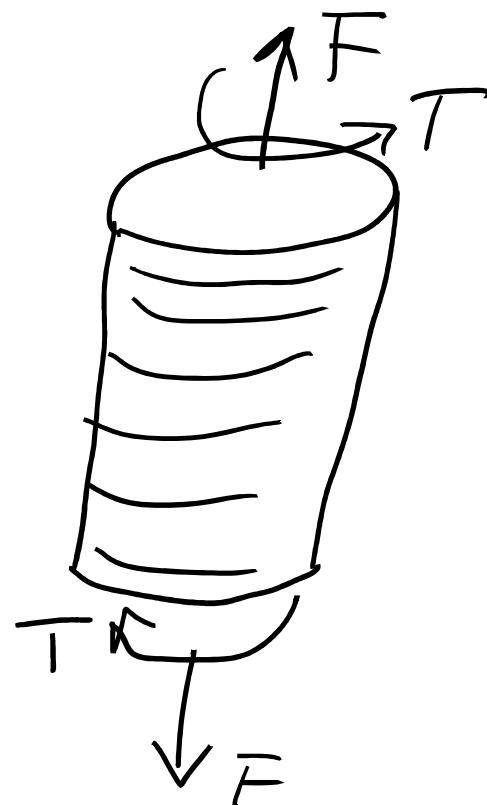
$\downarrow F$

$$\begin{pmatrix} \epsilon_x \\ \epsilon_y \\ 0 \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix}$$

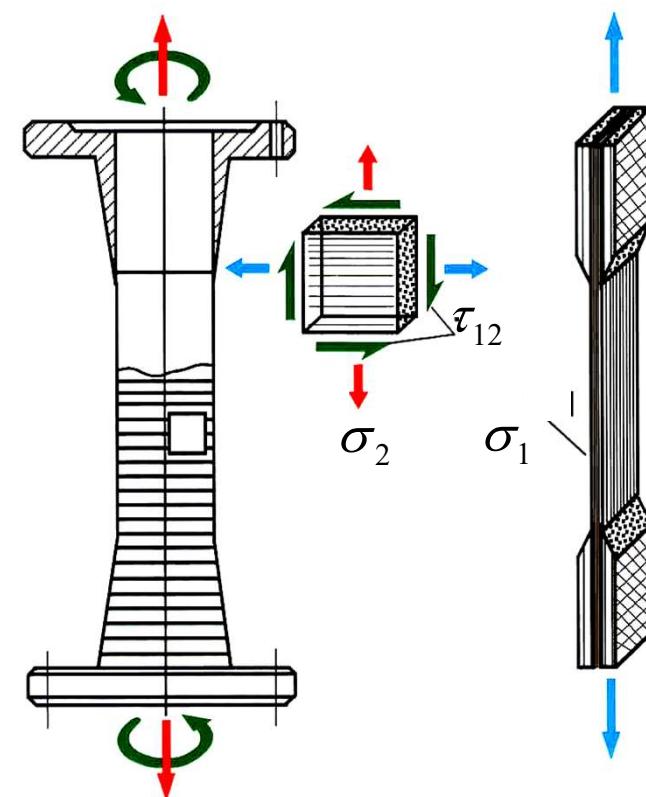
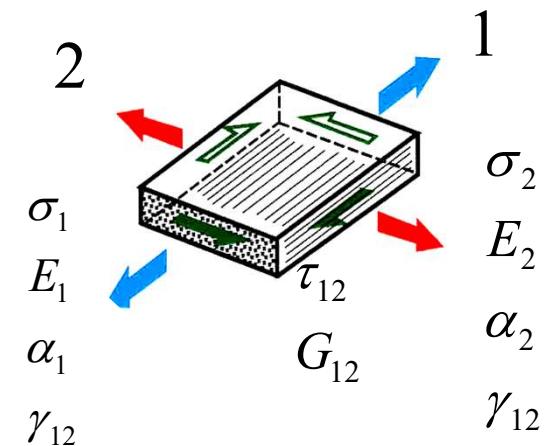
$$\begin{pmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \gamma_{12} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{pmatrix} \begin{pmatrix} \bar{\epsilon}_{11} \\ \bar{\epsilon}_{22} \\ \bar{\gamma}_{12} \end{pmatrix} \Rightarrow \gamma_{12} = S_{66} \bar{\gamma}_{12} \rightarrow G_{12}$$

⑤

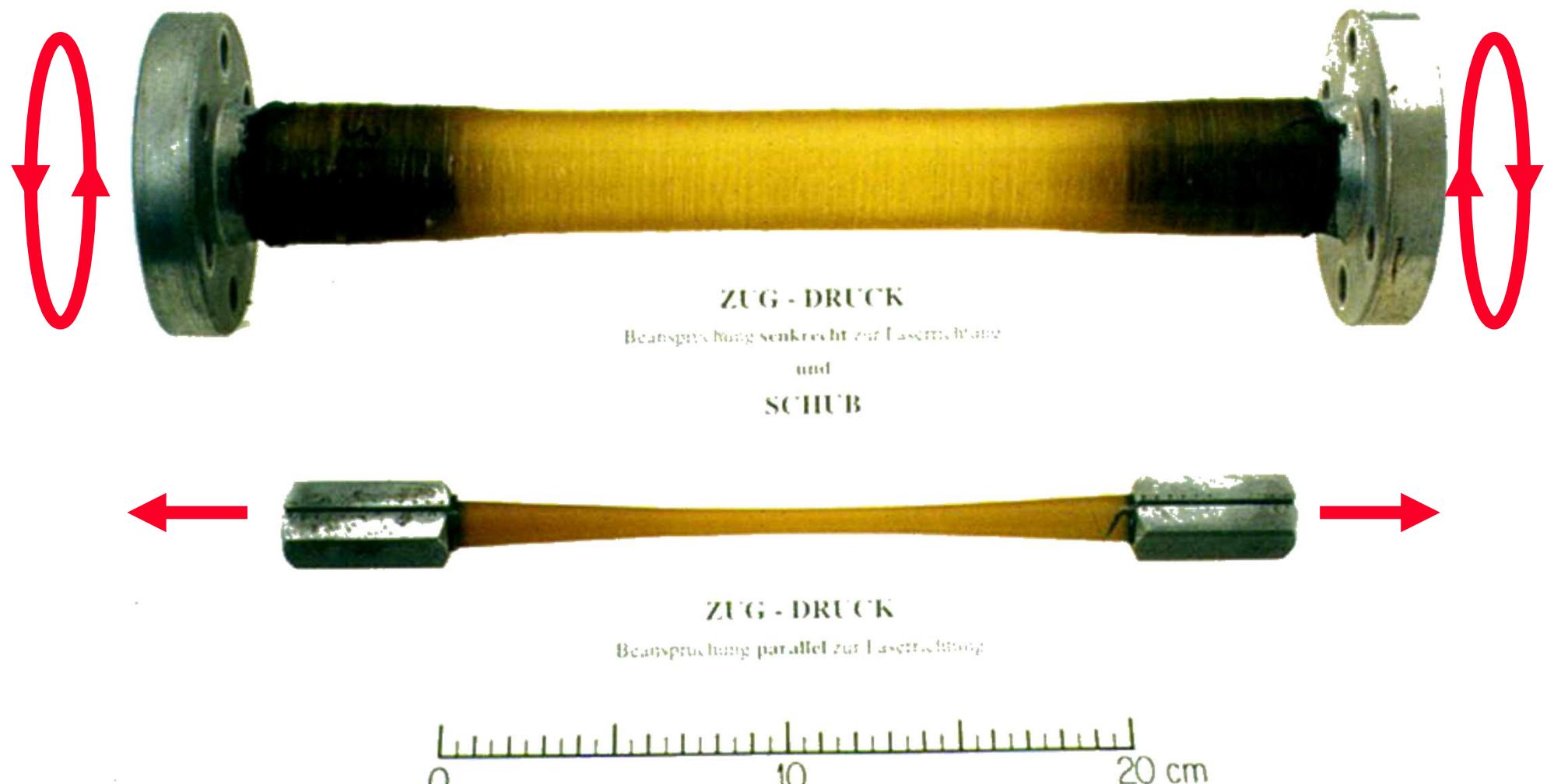
Biaxial tension/torsion test



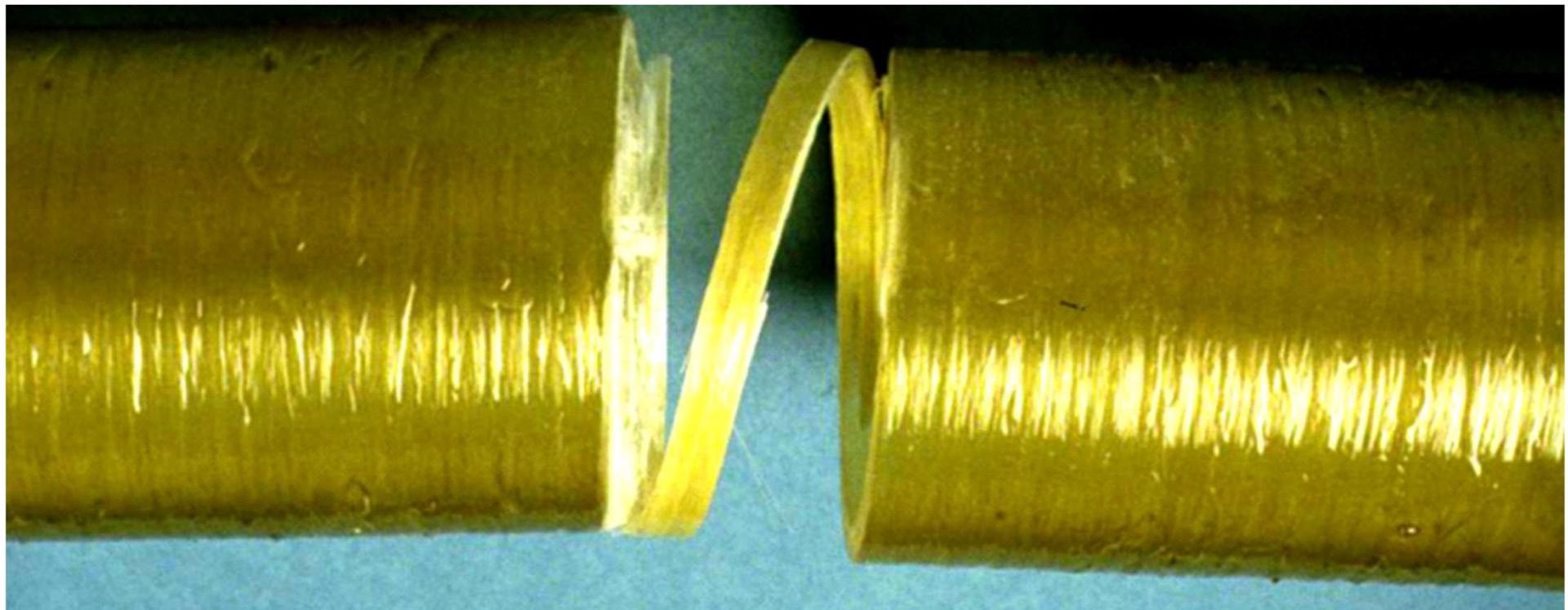
## Experimental determination of the UD-Lamina properties:



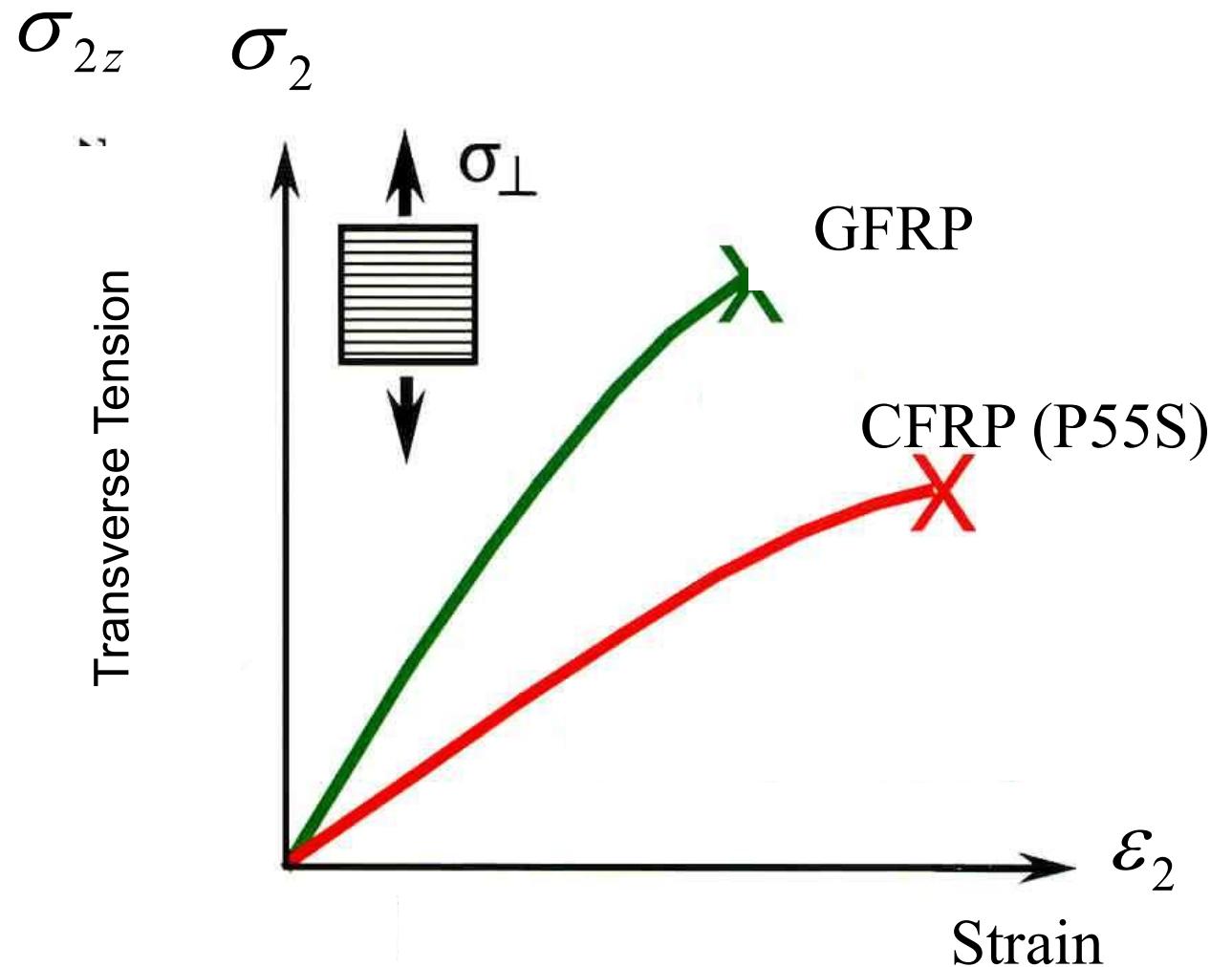
## Torsion and Tensile Samples



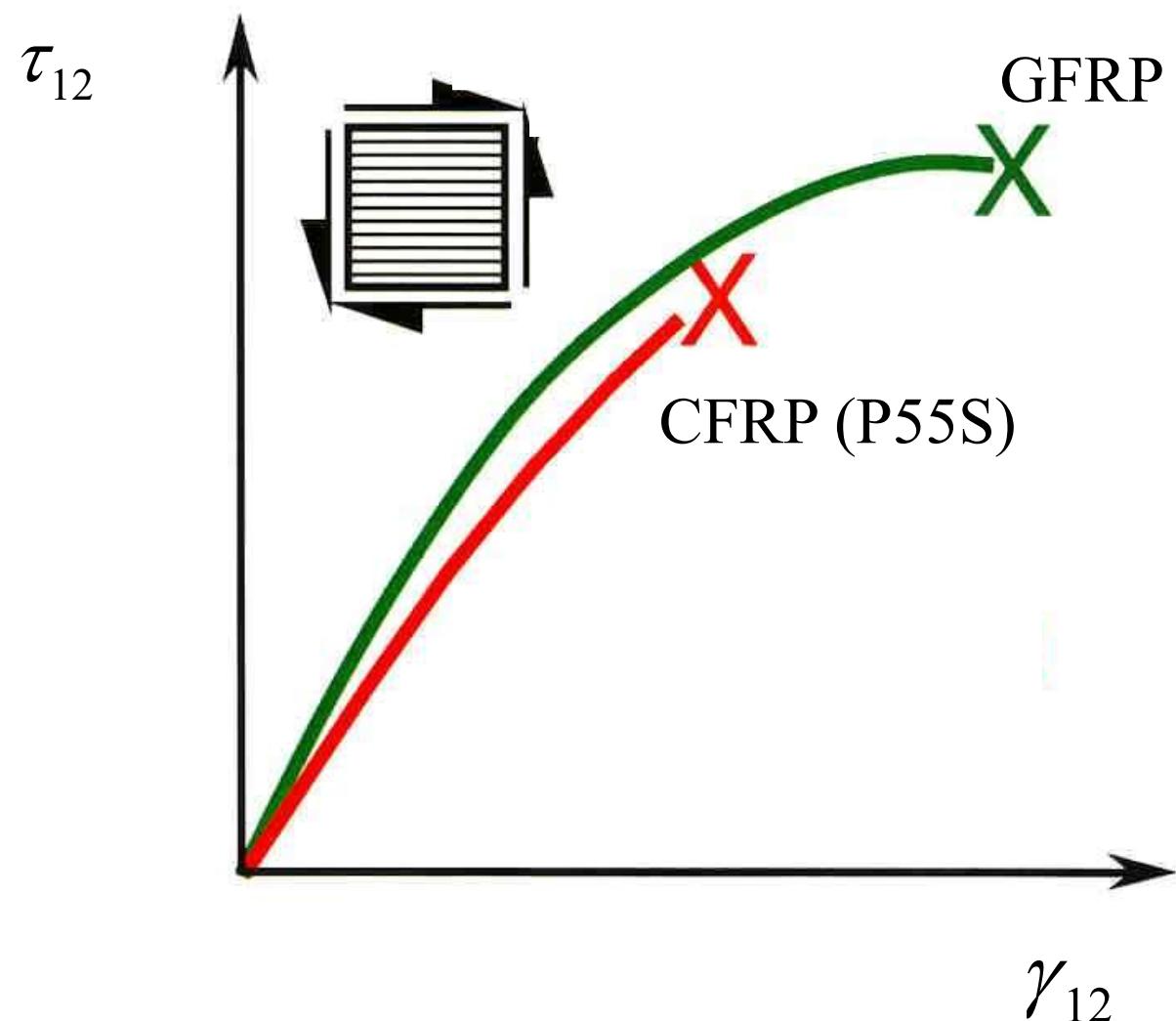
A torsion sample after the test



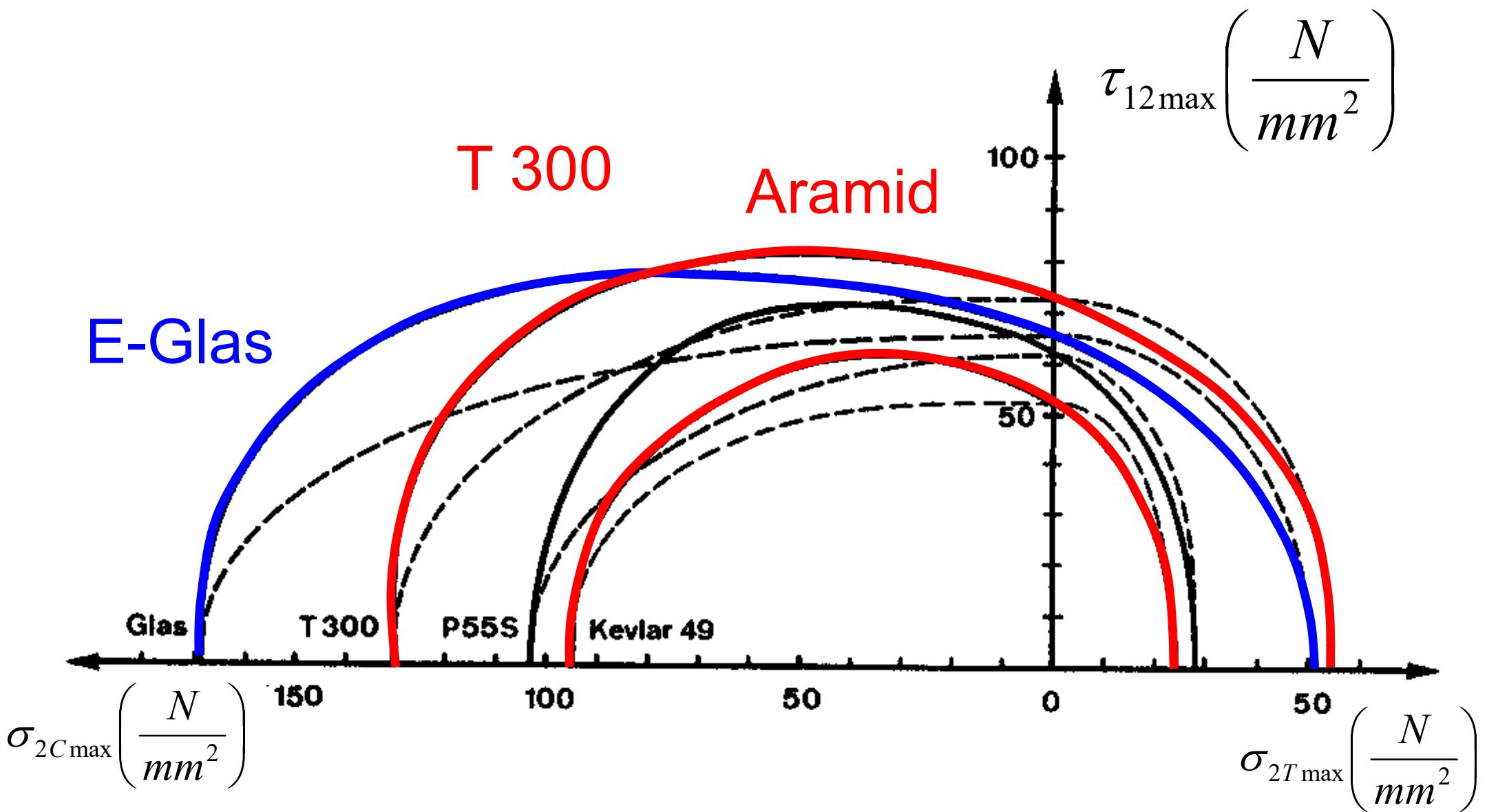
## Transverse Tension



# Shear



## Combined shear and transverse stresses



## Strength of some UD-Laminas

Lamina type	T 300/5208	B (4)/5505	AS/3501	Scotchply 1002	Kevlar 49 / Epoxy
$\sigma_{1T\max}$ (N/mm <sup>2</sup> )	1500	1260	1447	1062	1400
$\sigma_{1C\max}$ (N/mm <sup>2</sup> )	1500	2500	1447	610	235
$\sigma_{2T\max}$ (N/mm <sup>2</sup> )	40	61	51.7	31	12
$\sigma_{2C\max}$ (N/mm <sup>2</sup> )	246	202	206	118	53
$\tau_{12\max}$ (N/mm <sup>2</sup> )	68	67	93	72	34

## < Mechanics of a Lamina >

### List of symbols :

$\epsilon_{11}, \epsilon_{22}, \dots, \gamma_{23}, \gamma_{13}, \dots$  : normal and shear strains  
UD-Lamina; local directions

$\sigma_{11}, \sigma_{22}, \dots, \tau_{23}, \tau_{13}, \dots$  : normal and shear stresses  
UD-Lamina; local directions

$S_{11} = \frac{1}{E_{11}}$ ;  $\epsilon_{12} = -\frac{\nu_{12}}{E_{11}}, \dots$  : Components of Compliance matrix  
 $E_{11}, E_{22}, \dots, G_{12}, \dots$  E- and G-modulus  
in principal directions

$C_{11} = \frac{S_{22} S_{33} - S_{23} S_{32}}{S}; C_{12}, \dots$  : Components of stiffness matrix

$\nu_{12}, \nu_{13}, \nu_{23}$  : Poisson's ratios

$\begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}$  : stiffness matrix of a UD-Lamina

$\begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{12} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix}$  : Compliance matrix of a UD-Lamina

$E_F, \nu_F$  : fibre E-modulus and fibre Poisson's ratio

$E_M, \nu_F$  : matrix E-modulus and matrix Poisson's ratio

$\varphi$  (or  $\psi_f$  or  $V^f$ ) : fibre volume fraction

$E_1 = E_{11}; E_2 = E_{22}$  : UD-Elastic moduli

$\alpha_1; \alpha_2$  : UD Thermal expansion coefficients  
in local direction

$\alpha_x; \alpha_y, \alpha_{xy}$  : UD Thermal expansion Coefficients  
in global direction